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DEPARTMENT OF MATHEMATICS

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THE GENERAL HIGH INDICES THEOREM

Bo Johansson

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Introduction.

0.1. High indices theorem.

Suppose that

$$f(x) = \sum_1^\infty a_n e^{-p_n x}, \quad 0<p_1<p_2< \dots <p_n \to\infty, \ n\to\infty,$$

is convergent for $x>0$. If the series converges to the sum A for $x=0$, then $f(x) \to A$ as $x\to+0$, but the converse inference cannot be made without subsidiary hypotheses. Such hypotheses may be on the coefficients $\{a_n\}$, or on the exponents $\{p_n\}$, or on both. When the restrictions are on the coefficients, we have a Tauberian theorem, but if the implication holds when the exponents form a Hadamard sequence, we have a high indices theorem.

Littlewood conjectured a high indices theorem for Abel summation. This was proved in 1924 by Hardy and Littlewood [7]. The result is:

Theorem A. Suppose that

$$f(x) = \sum_1^\infty a_n e^{-p_n x}$$

converges for $x>0$, *where*

$$0<p_1<p_2< \dots \ ; \quad p_{n+1}/p_n \geq \theta >1 \ , \ n = 1,2, \dots .$$

If

$$\lim_{x\to+0} f(x) = A,$$

then

$$\sum_1^\infty a_n = A.$$

A simplified proof of this theorem was given by Ingham [9] in 1936. (See also Hardy [8], p. 182).

Theorem A can be stated in a different form. Let $\lambda_n = \log p_n$ and put $x = \exp(-\exp(-t))$, $t \in R$, then the theorem reads:

Theorem A'. Let $\{\lambda_n\}$ be a sequence of positive numbers such that

$$\lambda_{n+1} - \lambda_n \geq L > 0 \ , \ n = 1,2, \ldots .$$

Suppose that

$$F(t) = \sum_1^\infty a_n e^{-e^{-(t-\lambda_n)}} = \sum_1^\infty a_n \int_{\lambda_n - t}^\infty e^{-e^y + y} dy$$

converges for all real t. If

$$\lim_{t\to\infty} F(t) = A,$$

then

$$\sum_1^\infty a_n = A.$$

Theorem A remained an essentially isolated theorem until 1938 when Levinson [11] proved the following generalization of theorem A'.

Theorem B. Let

$$F(x) = \sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)\, dy = O(1)$$

converge uniformly for all $x \leq X$, for any $X < \infty$. Let

$$\lambda_{n+1} - \lambda_n \geq L > 0 \ , \ n = 1,2, \ldots \quad (\lambda_1 > 0).$$

Let $K \in L^1(\mathbb{R})$. Its Fourier transform is

$$k(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty K(x)\, e^{-iux} dx.$$

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Let $k(u+iv)$ *be an analytic function of* $w = u+iv$ *in the halfplane* $v > 0$ *and continuous for* $v \geq 0$, *and let* $k'(u)$ *exist*. *Let* $c = 2\pi/L$ *and let* A *be a positive constant*. *Suppose* ξ *is a real variable*. *Let*

(1) $$\max_{|\xi|\leq c} \left|\frac{k(u+\xi)}{k(u)}\right| \leq e^{\Theta(u)} ,$$

(2) $$\max_{|\xi|\leq c} \left|\frac{k(w+\xi)}{k(w)}\right| \leq A\, e^{A|w|} , \qquad v \geq 0 ,$$

(3) $$\left|\frac{k'(u)}{k(u)}\right| \leq e^{\Theta(u)} ,$$

where $\Theta(u)$ *is a positive even function of* u, *monotonely increasing for* $u > 0$, *and*

$$\int_1^\infty \frac{\Theta(u)}{u^2}\, du < \infty .$$

Then

$$\limsup_{n\to\infty} |a_n| \leq C_0 \limsup_{x\to\infty} F(x) ,$$

where C_0 *is a constant depending only on* $K(x)$ *and* L.

Theorem B'. *Let the hypotheses of theorem B be satisfied*. *In addition let* $xK(x) \in L^1(\mathbb{R})$. *Then*

$$\lim_{x\to\infty} F(x) = 0$$

implies that

$$\sum_1^\infty a_n = 0 .$$

The assumptions in theorem B' are fulfilled by a wide class of kernels, and in particular, the kernel in theorem A', with $k(u) = \frac{1}{\sqrt{2\pi}}\,\Gamma(1-iu)$, belongs to this class. It also follows that the high indices theorem holds for a Lambert series.

A series

$$\sum_{1}^{\infty} a_n \frac{x^{p_n} \log(1/x)^{p_n}}{1 - x^{p_n}} \quad , \quad x \in (0,1) \ ,$$

is called a Lambert series, and in this case

$$k(u) = \frac{1}{\sqrt{2\pi}} \ iu \ \Gamma(1-iu) \ \zeta(1-iu) \quad .$$

That $k(u+iv)$ satisfies (1), (2), and (3) follows from well-known theorems about the Γ- and ζ-functions.

0.2. Rényi's conjecture.

Several people have been interested in the problem whether there is a high indices theorem for the series

$$\text{(4)} \qquad \sum_{1}^{\infty} a_n \frac{x^{p_n}}{1+x^{p_n}} \quad , \quad x \in [0,1) \quad .$$

For this kernel the hypotheses in Levinson's theorem are not fulfilled, but Rényi [15] conjectured in 1959 that there is a high indices theorem if $p_n = 2^n$. This conjecture was proved by Halász [6] in 1967. In 1969 Korevaar [10] observed that there is no high indices theorem for a certain Hadamard sequence of exponents.

Thus it is an open problem to characterize the exponents for which there is a high indices theorem for the series (4). That problem will be discussed in the first two chapters of this thesis. After some preliminary remarks on "poor-approximability", we prove a generalization of Levinson's theorem B', which is applicable to the Rényi-kernel. The Fourier transform of the kernel is now allowed to have a great number of zeros in the upper halfplane, and we show that the admissible sequences of exponents are related to these zeros.

0.3. On the absolute summability of series.

Let

(5) $$f(x) = \sum_{1}^{\infty} a_n e^{-p_n x}, \quad x>0,$$

be a lacunary series, i.e. $p_{n+1}/p_n \geq \Theta > 1$, $n = 1,2, \ldots$ $(p_1 > 0)$.

If $f' \in L^1(0,\infty)$, then $\lim_{x \to +0} f(x)$ exists, and it follows from theorem A that the series in (5) is convergent for x=0. However, in addition Zygmund [20] has proved:

Theorem C. If the series in (5) is lacunary and $f' \in L^1(0,\infty)$, *then the series is absolutely convergent at* x=0. *Moreover,*

$$\sum_{1}^{\infty} |a_n| \leq A \int_{0}^{\infty} |f'(x)|\, dx ,$$

where A *is a constant depending only on* Θ.

The proof given by Zygmund uses the main idea in Ingham's proof of theorem A. Later Waterman [18] refined the same idea and proved:

Theorem D. If the series in (5) is lacunary, $m > 1$, *and*

$$\int_{0}^{\infty} |f'(x)|^m\, dx < \infty ,$$

then

$$\sum_{1}^{\infty} |a_n|^m p_n^{m-1} \leq A \int_{0}^{\infty} |f'(x)|^m dx ,$$

where A *is a constant depending on* m *and* Θ *only.*

In Chapter 3 we generalize these results to a large class of kernels.

0.4. The remainder in the high indices theorem.

Remainder problems have also been considered in connection with the high indices theorem, i.e. if

(6) $$f(x) = \sum_{1}^{\infty} a_n e^{-p_n x} , \quad x>0,$$

then we want an estimate of the coefficients when the rate of increase (decrease) of the function f is known as $x \to +0$. A result in this direction given by Gaier [3] reads:

Theorem E. If

$$\sum_{1}^{\infty} a_n e^{-p_n x} = O(x^{-\alpha}) \quad \text{as} \quad x \to +0 \quad , \quad \alpha > 0 \quad ,$$

is a lacunary series, then

$$a_n = \begin{cases} O(p_n^{\alpha} (\log p_n)^{1-\alpha}) & , n \to \infty \quad ; \quad 0 < \alpha < 1 \\ O(p_n \log n) & , n \to \infty \quad ; \quad \alpha = 1 \\ O(p_n^{\alpha}) & , n \to \infty \quad ; \quad \alpha > 1 \end{cases}$$

There are also results for decreasing f. (See Ganelius [4], theorem 8.2; Halász [5]).

Theorem F. Let

$$\sum_{1}^{\infty} a_n e^{-p_n x} = O(x^{\alpha}) \quad \text{as} \quad x \to +0 \quad , \alpha>0 \quad ,$$

be a lacunary series. Then

$$a_n = O(p_n^{-\alpha}) \quad \text{as} \quad n \to \infty .$$

Sharp variants of Gaier's result will be given in Chapter 4, where we also extend the theorems E and F to general kernels.

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Chapter 1.

1.1. Introduction.

In this chapter we will give some preliminary results and as an easy consequence prove theorem A. This proof is very special, because it depends on the homogeneity property of the powers of x and can not be used for general kernels.

The method is based on Korevaar's notation of "poor approximability" (see [10]), a term which we define in the following way:

Definition. Let B *be a normed space and let* $\{f_k\}_1^\infty$ *be a sequence in* B. *If there exists a positive number* d *such that*

$$\inf_{n\geq 1} \inf_{\{b_k\}} \| f_n - \sum_{k\neq n} b_k f_k \|_B = d,$$

then $\{f_k\}_1^\infty$ *is called a family of poor approximability in* B.

We also use the following notation: If k is a function, then $\check{k}$ is the function defined by $\check{k}(x) = k(-x)$, $x \in \mathbb{R}$, and if $k \in L^1(\mathbb{R})$, then its Fourier transform $\hat{k}$ is defined by

$$\hat{k}(u) = \int_{-\infty}^{\infty} k(x) e^{-ixu}\, dx.$$

1.2. A theorem about poor approximability.

We will now formulate a theorem about poor approximability in $L^\infty(\mathbb{R})$, where the family consists of translates of a function.

Unspecified signs of integration will always denote integration over the whole real line.

Theorem 1.1. Let $f \in AC(\mathbb{R})$ *and suppose that*

(1.2.1) $$\int e^{-ixu}\, df(x) \neq 0 \quad \textit{for} \quad u \in \mathbb{R}.$$

Let $\{\lambda_k\}_{-\infty}^\infty$ *be a sequence of real numbers, which are separated by a positive constant* c, *i.e.*

(1.2.2) $$\lambda_{k+1} - \lambda_k \geq c > 0.$$

Then $\{f(\cdot-\lambda_k)\}_{-\infty}^{\infty}$ is a family of poor approximability in $L^\infty(\mathbb{R})$.

To prove theorem 1.1, we need an interpolation result. This will be stated in the next lemma. (A weaker result can be found in Levinson [12]). But first we will introduce the customary notation in distribution theory. Thus, $\mathcal{S}$ is the set of rapidly decreasing C^∞-functions and $\mathcal{S}'$ is the set of tempered distributions. $\mathcal{D}$ stands for the set of test functions, i.e. C^∞-functions with compact support and $\mathcal{D}'$ for the set of distributions. (Cf Schwartz [17]).

Definitions. Let $u \in \mathcal{S}'$ and $\phi \in \mathcal{S}$. Then $\tau_x \phi$ $(x \in \mathbb{R})$ and $u * \phi$ are the functions defined by

$$\tau_x\phi(y) = \phi(y-x), \quad y \in \mathbb{R},$$

and

$$u * \phi(y) = u(\tau_y \check{\phi}), \quad y \in \mathbb{R}.$$

The Fourier transform $\hat{u}$ of u is defined by

$$\hat{u}(\phi) = u(\hat{\phi}) \quad \text{for all} \quad \phi \in \mathcal{S}.$$

If $\phi \in \mathcal{D}$, then the norms of ϕ are denoted by

$$\|\phi\|_N = \max\{|D^k \phi(x)|;\ x \in \mathbb{R},\ k \leq N\}.$$

Lemma 1.2. Let $\{\lambda_n\}$ be a sequence of real numbers satisfying

$$\lambda_{n+1} - \lambda_n \geq c > 0, \quad n \in \mathbb{Z}.$$

Then, for each integer n and each $\varepsilon > 0$, there exists a function $H_n \in \mathcal{S}$ such that

(1.2.3) $$H_n(\lambda_n) = 1; \quad H_n(\lambda_k) = 0, \quad k \neq n,$$

and

(1.2.4) $$\operatorname{supp}(\hat{H}_n) \subset [-\pi/c - \varepsilon, \pi/c + \varepsilon].$$

If

(1.2.5) $$\hat{G}_n(u) = \hat{H}_n(u)\, e^{i\lambda_n u},$$

then

(1.2.6) $$\|\hat{G}_n\|_N \leq C_N < \infty \quad \text{(independent of } n).$$

Proof. Keep n fixed and let $k \in \mathbb{Z}\setminus\{0\}$. Define a sequence $\{x_k\}$ by

$$x_k = \begin{cases} kc, \text{ if } |kc - \lambda_m + \lambda_n| > c/2 \text{ for all } m \\ kc, \text{ if } kc - \lambda_m + \lambda_n = c/2 \text{ for some } m \\ \lambda_m - \lambda_n, \text{ if } -c/2 \leq kc - \lambda_m + \lambda_n < c/2 \text{ for some } m. \end{cases}$$

This determines the sequence $\{x_k\}$ uniquely. Moreover,

(1.2.7) $$x_{k+1} - x_k \geq c/2$$

and

(1.2.8) $$|x_k - kc| \leq c/2.$$

Consider the function

(1.2.9) $$T_n(z) = \prod_1^\infty \left(1 - \frac{z}{x_k}\right)\left(1 - \frac{z}{x_{-k}}\right)$$

and estimate

(1.2.10) $$\left|\frac{T_n(z)}{\sin \frac{\pi z}{c}}\right| = \frac{c}{\pi|z|} \prod_1^\infty \left|\frac{1-z/x_k}{1-z/kc}\right|\left|\frac{1-z/x_{-k}}{1+z/kc}\right|.$$

A term in the infinite product (1.2.10) can be estimated by

(1.2.11) $$\left|\frac{1-z/x_k}{1-z/kc}\right| = \left|1 + \frac{z/kc - z/x_k}{1 - z/kc}\right| \leq 1 + \frac{|z|\,|x_k-kc|}{|x_k|\,|z-kc|} \leq$$

$$\leq 1 + \frac{|z|\ c/2}{(|k|c - c/2)|z - kc|} \leq 1 + \frac{|z|}{|k|\ |z-kc|} \leq \exp\left\{\frac{|z|}{|k|\,|z-kc|}\right\},$$

where (1.2.7) and (1.2.8) have been used.

Let $\operatorname{Re} z \geq 0$ and $Nc - c/2 \leq |z| < Nc - c/2$, $N \geq 5$. Then we find that

$$(1.2.12)\qquad \sum_{k\neq 0,\pm N} \frac{1}{|k|\,|z-kc|} \leq \frac{1}{c}\sum_{-\infty}^{-N-1}\frac{1}{k^2} + \frac{1}{|z|}\sum_{-N+1}^{-1}\frac{1}{|k|} + \frac{3}{|z|}\sum_{1}^{[N/2]}\frac{1}{k} +$$

$$+ \frac{2}{Nc}\sum_{[N/2]+1}^{N-1}\frac{1}{N-k-1/2} + \frac{1}{Nc}\sum_{N+1}^{2N}\frac{1}{k-N-1/2} + \frac{1}{c}\sum_{2N+1}^{\infty}\frac{1}{(k-N-1/2)^2} \leq$$

$$\leq \frac{12}{|z|} + \frac{10\log|z|}{|z|} - \frac{10\log c}{|z|}.$$

Inserting (1.2.11) and (1.2.12) in (1.2.10), we get

$$(1.2.13)\qquad \left|\frac{T_n(z)}{\sin\frac{\pi z}{c}}\right| \leq A_c|z|^9 \left|\frac{1-z/x_N}{1-z/Nc}\right| \left|\frac{1-z/x_{-N}}{1+z/Nc}\right|.$$

The inequality (1.2.13) is obviously true in the whole annulus $Nc - c/2 \leq$ $\leq |z| \leq Nc + c/2$, and therefore the inequality

$$|T_n(z)| \leq A(1+|z|^9)\exp(\tfrac{\pi}{c}|\operatorname{Im} z|)$$

holds for all $z \in \mathbb{C}$. By a generalization of a well-known theorem of Paley and Wiener (Rudin [16], theorem 7.23), we have

$$T_n \in \mathcal{S}' \quad\text{and}\quad \operatorname{supp}\ \hat{T}_n \subset [-\pi/c, \pi/c].$$

Let ϕ be a function in $\mathcal{S}$ with $\operatorname{supp}\hat{\phi} \subset (-\varepsilon,\varepsilon)$ and $\phi(0) = 1$. Putting $G_n =$ $= \phi\cdot T_n$, we get, for each n, a function in $\mathcal{S}$ with Fourier transform.

$$\hat{G}_n = \hat{T}_n * \hat{\phi} \quad\text{and}\quad \operatorname{supp}(\hat{G}_n) \subset [-\tfrac{\pi}{c} - \varepsilon,\ \tfrac{\pi}{c} + \varepsilon].$$

These functions satisfy (1.2.6) with constants C_N, $N = 1, 2,\ldots,$ independent of n. If H_n is defined by

$$H_n(z) = G_n(z-\lambda_n),\quad z \in \mathbb{C},$$

then the condition (1.2.3) follows from (1.2.9) and the conditions (1.2.4) and (1.2.5) are easily seen to be valid.

The technique in the proof of theorem 1.1 was developed in joint work with Faxén (see also [2]).

Proof of theorem 1.1.

For each integer n, we want to construct a measure ν_n, which is orthogonal to the span $\{f(\cdot-\lambda_k)\}_{k\neq n}$ and satisfies $|\nu_n(f(\cdot-\lambda_n))| \geq \delta > 0$,

where

$$\nu_n(g) = \int g(x)d\nu_n(x) \quad \text{and} \quad \|\nu_n\| = 1.$$

This can be done in the following way:

From Wiener's lemma (see [4], p. 11, or [8], p. 290) and because of (1.2.1) it follows that there is a $\psi \in L^1(\mathbb{R})$ such that

(1.2.14) $$\frac{1}{(f')^\wedge(-u)} = \hat{\psi}(u) \quad \text{for} \quad u \in [-2\pi/c, 2\pi/c].$$

Defining

(1.2.15) $$\mu_n = H_n * \psi,$$

we get the estimate

(1.2.16) $$\|\mu_n'\|_{L^1} \leq \|H_n'\|_{L^1}\|\psi\|_{L^1} \leq C_1\|H_n\|_{L^1}\|\psi\|_{L^1} \leq C_1 C_2\|\psi\|_{L^1}$$

for the derivative of μ_n. The first inequality comes from (1.2.15). The second inequality is Bernstein's, which is applicable because of (1.2.4), and the third is obvious from the construction of the functions $\{H_n\}$. Thus,

$$\int f(x-\lambda)d\mu_n(x) = -\int f'(x-\lambda)\mu_n(x)dx = -\int (f')^\wedge(-u)\,\hat{\mu}_n(u)e^{iu\lambda}du =$$
$$= -\int \hat{H}_n(u)e^{iu\lambda}du = -2\pi H_n(\lambda), \quad \lambda \in \mathbb{R},$$

where we have applied (1.2.14) and (1.2.15).

In particular,

(1.2.17) $$\frac{1}{2\pi}\int f(x-\lambda_n)d\mu_n(x) = -1; \; \frac{1}{2\pi}\int f(x-\lambda_k)d\mu_n(x) = 0, \quad k \neq n.$$

From (1.2.16) and (1.2.17) it follows that

$$\inf_{\{b_k\}} \|f(\cdot-\lambda_n) - \sum_{k\neq n} b_k f(\cdot-\lambda_k)\|_{L^\infty(\mathbb{R})} =$$
$$= \max_{\substack{\Lambda_n \in (L^\infty)^* \\ \|\Lambda_n\| = 1 \\ \Lambda_n|\operatorname{span}\{f(\cdot-\lambda_k)\}_{k\neq n} = 0}} |\Lambda_n(f(\cdot-\lambda_n))| \geq \frac{|\mu_n(f(\cdot-\lambda_n))|}{\|\mu_n'\|_{L^1(\mathbb{R})}} \geq \frac{2\pi}{C_1\cdot C_2\|\psi\|_{L^1(\mathbb{R})}}$$

and the theorem is proved.

Assumption (1.2.2) in the previous theorem can not be weakened, for if f is absolutely continuous and

$$\inf_{n\geq 1} (\lambda_{n+1}-\lambda_n) = 0,$$

then

$$c_n = \max_{\substack{\Lambda\in(L^\infty)^* \\ \|\Lambda\|=1 \\ \Lambda|\mathrm{span}\{f(\cdot-\lambda_k)\}_{k\neq n} = 0}} |\Lambda(f(\cdot-\lambda_n)| = |\Lambda_n(f(\cdot-\lambda_n)| =$$

$$= |\Lambda_n(f(\cdot-\lambda_n) - f(\cdot-\lambda_{n-1}))| \leq \|f-f(\cdot-(\lambda_n-\lambda_{n-1}))\|_{L^\infty(\mathbb{R})},$$

but this contradicts that the constants c_n, $n = 1,2,\ldots$, have a positive lower bound.

1.3. A Tauberian result for distributions.

In this section we will treat a very special class of tempered distribution. We formulate the result as

Theorem 1.3. Let $\{\lambda_n\}_1^\infty$ *be a sequence of positive numbers such that*

(1.3.1) $$\lambda_{n+1} - \lambda_n \geq c > 0.$$

Suppose the coefficients $\{a_n\}_1^\infty$ *satisfy*

$$a_n = O(\lambda_n^N) \quad \textit{for some finite constant } N.$$

Suppose further that $k \in \mathcal{S}$ *and that there exists a closed interval* I *of length* $2\pi/c$ *such that*

(1.3.2) $$\hat{k}|_I \neq 0 \quad \textit{and} \quad \hat{k}(0) \neq 0.$$

If

$$\sum_1^\infty a_n \int_{\lambda_n-x}^\infty k(y)dy = O(1) \quad (x\in\mathbb{R}),$$

then

(1.3.3) $$\sum_{n\leq x} a_n = O(1).$$

In particular, if

$$\lim_{x\to\infty} \sum_1^\infty a_n \int_{\lambda_n - x}^\infty k(y)dy = 0, \tag{1.3.4}$$

then

$$\sum_1^\infty a_n = 0. \tag{1.3.5}$$

Remark. The theorem is still true if the hypothesis (1.3.1) is replaced by $\liminf_{n\to\infty} (\lambda_{n+1}-\lambda_n) = c > 0$.

Proof. Let

$$F(x) = \sum_1^\infty a_n \int_{\lambda_n - x}^\infty k(y)dy.$$

Then we have

$$F'(x) = \mu * k(-x),$$

where $\mu = \sum_1^\infty a_n \tau_{-\lambda_n}\delta$ is in $\mathcal{S}'$. (δ is the Dirac measure). Without loss of generality we can assume that $I = [-\pi/c, \pi/c]$. Since (1.3.2) holds, it exists a $\beta > \pi/c$ such that $\hat{k}(u) \neq 0$, $u \in [-\beta, \beta]$. Thus there exists a $\psi \in \mathcal{S}$ satisfying

$$\hat{\psi}(u) = \frac{1}{\hat{k}(u)}, \quad u \in [-\beta, \beta].$$

Let $\{H_k\}$ be the functions given in lemma 1.2 with $\operatorname{supp}(\hat{H}_k) \subset (-\beta, \beta)$, we get

$$((F')^{\vee} * \psi * H_k)^{\wedge} = \hat{\psi} \cdot \hat{H}_k \cdot \hat{k}\, \hat{\mu} = (\mu * H_k)^{\wedge} \quad \text{in } \mathcal{S}'.$$

Hence,

$$\mu * H_k = (F')^{\vee} * \psi * H_k = -\check{F} * \psi' * H_k.$$

Because F is bounded, we have

$$|a_k| = |\mu * H_k(0)| = \left|\int F(y)\ \psi' * H_k(y)dy\right| \leq \|F\|_{L^\infty} \|\psi'\|_{L^1} \|H_k\|_{L^1}. \tag{1.3.6}$$

If (1.3.4) holds, we get a better estimate than (1.3.6) by

$$|a_k| \le \sup_{y>\lambda_k/2} |F(y)| \int_{y>\lambda_k/2} |\psi' * H_k(y)|dy +$$

$$+ \sup_{y<\lambda_k/2} |F(y)| \int_{y<\lambda_k/2} |\psi' * H_k(y)|dy.$$

The second integral to the right can be estimated by

$$\int_{y<\lambda_k/2} |\psi' * H_k(y)|dy \le \int |\psi'(t)|dt \int_{y<\lambda_k/2} |H_k(y-t)|dy =$$

$$= \int |\psi'(t)|dt \int_{y<-\lambda_k/2-t} |G_k(y)|dy \le$$

$$\le \int_{t>-\lambda_k/4} |\psi'(t)|dt \int_{y<-\lambda_k/4} |G_k(y)|dy + \int_{t<-\lambda_k/4} |\psi'(t)|dt \int |G_k(y)|dy \le$$

$$\le \|\psi'\|_{L^1} \int_{y<-\lambda_k/4} |G_k(y)|dy + \|G_k\|_{L^1} \int_{t<-\lambda_k/4} |\psi'(t)|dt,$$

where G_k is the function in lemma 1.2. Hence we find that

$$\lim_{k\to\infty} |a_k| = 0. \tag{1.3.7}$$

In order to get (1.3.3) and (1.3.5), let $x_N = (\lambda_N + \lambda_{N+1})/2$ and suppose that $\hat{k}(0) = 1$. Then

$$\left|\sum_1^N a_n - f(x_N)\right| = \left|\sum_1^N a_n \int_{-\infty}^{\lambda_n - x_N} k(y)dy + \sum_{N+1}^{\infty} a_n \int_{\lambda_n - x_N}^{\infty} k(y)dy\right|. \tag{1.3.8}$$

Now (1.3.3) follows readily from (1.3.6) and (1.3.8). The conclusion (1.3.5) follows from (1.3.7) and (1.3.8), because

$$\left|\sum_1^N a_n - f(x_N)\right| \le \sum_1^{[N/2]} |a_n| \int_{-\infty}^{\lambda_n - x_N} |k(y)|dy + \sum_{[N/2]}^{N} |a_n| \int_{-\infty}^{\lambda_n - x_N} |k(y)|dy +$$

$$+ \sum_{N+1}^{\infty} |a_n| \int_{\lambda_n - x_N}^{\infty} |k(y)|dy \le C_1 \max |a_n| \sum_1^{[N/2]} \frac{1}{1+(\lambda_n - x_N)^2} +$$

$$+ C_1 \max_{n\ge[N/2]} |a_n| \sum_{[N/2]}^{N} \frac{1}{1+(\lambda_n - x_N)^2} + C_1 \max_{n\ge N+1} |a_n| \sum_{N+1}^{\infty} \frac{1}{1+(\lambda_n - x_N)^2} \le$$

$$\leq C_1 \max|a_n| \sum_{1}^{[N/2]} \frac{1}{1+(N-n)^2 c^2} + C_2 \max_{n\geq[N/2]} |a_n| \leq$$

$$\leq C_3 \sum_{[N/2]}^{N} \frac{1}{1+c^2 \cdot n^2} + C_2 \max_{n\geq[N/2]} |a_n| ,$$

where C_1, C_2 and C_3 are constants.

1.4. Proof of theorem A.

We conclude by giving a proof of Hardy-Littlewood's high indices theorem, using the methods of this chapter.

Suppose that

(1.4.1) $$\lim_{x\uparrow 1} \sum_{1}^{\infty} a_n x^{p_n} = 0, \quad p_{n+1}/p_n \geq \delta > 1.$$

Let $\eta < 1$. Then the series

(1.4.2) $$\sum_{1}^{\infty} a_n \eta^{p_n} x^{p_n}$$

converges uniformly on the interval $[0,1]$. If we transform (1.4.2) to the real axis by

$$x = \exp(-\exp(-t)) = k(t),$$

then the series

$$\sum_{1}^{\infty} a_n \eta^{p_n} k(t-\log p_n), \quad t \in \mathbb{R},$$

converges uniformly on $\mathbb{R}$.

The Fourier transform of k' can easily be calculated. We find that

(1.4.3) $$(k')^{\wedge}(u) = \Gamma(1+iu) \neq 0, \quad u \in \mathbb{R}.$$

The sequence $\{\log p_n\}$ satisfies

(1.4.4) $$\log p_{n+1} - \log p_n \geq \log \delta > 0.$$

Thus, theorem 1.1 states that the sequence $\{k(\cdot-\log p_n)\}_1^{\infty}$ is a family of poor approximability in $L^{\infty}(\mathbb{R})$, i.e. there exists a positive constant d such that

$$d \leq \inf_{\{b_m\}} \|k(\cdot-\log p_n) - \sum_{m\neq n} b_m k(\cdot-\log p_m)\|_{L^{\infty}} \leq$$

$$\leq \left\| k(\cdot - \log p_n) - \sum_{m \neq n} \frac{a_m \eta^{p_m}}{a_n \eta^{p_n}} k(\cdot - \log p_m) \right\|_{L^\infty} \qquad \text{if} \quad a_n \neq 0.$$

Hence,

$$d\eta^{p_n} |a_n| \leq \left\| \sum_1^\infty a_m \eta^{p_m} k(\cdot - \log p_m) \right\|_{L^\infty} = O(1).$$

Let $\eta \uparrow 1$, and we find that

(1.4.5) $\qquad a_n = O(1).$

By a simple transformation (1.4.1) can be written in the form (1.3.4). Since (1.4.3), (1.4.4) and (1.4.5) hold, theorem 1.3 can be applied and we get

$$\sum_1^\infty a_n = 0.$$

If the existence of $\lim\limits_{x\to\infty} \sum\limits_1^\infty a_n\, k(x-\lambda_n)$ is assumed, poor approximability of the family $\{k(\cdot-\lambda_n)\}$ is not sufficient for the validity of a high indices theorem as the next example shows.

<u>Example</u>. Let $f(x) = x - 2x^2$. Then we have

(1.4.6) $$\sum_0^\infty 2^n f(x^{2^n}) = x, \qquad x \in [0,1).$$

If (1.4.6) is transformed to the real axis by $x = \exp(-\exp(-t))$, $t \in \mathbb{R}$, then

(1.4.7) $$\sum_0^\infty 2^n k(t - n \cdot \log 2) = \exp(-\exp(-t)),$$

where $k(t) = \exp(-\exp(-t)) - 2\exp(-2\exp(-t))$. The Fourier transform of k' is

$$(k')\hat{}(u) = \Gamma(1+iu)\,(1-2^{1-iu}) \neq 0 \quad \text{for} \quad u \in \mathbb{R}.$$

It is now easily seen that $\{k(\cdot - n\log 2)\}$ is a family of poor approximability in $L^\infty(\mathbb{R})$, and from (1.4.7) it follows that

$$\lim_{t\to\infty} \sum_0^\infty 2^n k(t - n\cdot\log 2) = 1,$$

but

$$2^n \not\to 0 \quad \text{as} \quad n \to \infty.$$

Chapter 2

2.1. Introduction.

In this chapter we will give a generalization of theorem B'. We consider the following problem:

Assume that

$$(2.1.1) \qquad F(x) = \sum_{1}^{\infty} a_n \int_{\lambda_n - x}^{\infty} k(y)dy$$

converges uniformly on every set $\{x \in \mathbb{R} \, ; \, x < x_0\}$ and defines a bounded function on $\mathbb{R}$. Let $\{\lambda_n\}_1^{\infty}$ be a given sequence of positive numbers, which are well separated, i.e.

$$(2.1.2) \qquad \lambda_{n+1} - \lambda_n \geq c > 0.$$

If

$$\lim_{x \to \infty} F(x) \quad \text{exists,}$$

what can then be said about the convergence of $\sum_{1}^{\infty} a_n$?

A necessary condition in theorem B' is that the Fourier transform $\hat{k}$, of the kernel k, has an analytic continuation in the upper halfplane and is free from zeros there. What can happen, if the extension of the Fourier transform has zeros in this halfplane?

Let us give an example of a kernel of special interest. This is the Rényi kernel, which is given by $f(x) = x/(1+x)$ for $0 \leq x \leq 1$.

Does the existence of

$$(2.1.3) \qquad \lim_{x \uparrow 1} \sum_{1}^{\infty} a_n f(x^{p_n})$$

imply that

$$(2.1.4) \qquad \sum_{1}^{\infty} a_n \quad \text{is convergent} \quad ,$$

if $p_{n+1}/p_n \geq \Theta > 1$ $(p_1 > 0)$ and no other restrictions are imposed on the sequence $\{p_n\}_1^\infty$?

If (2.1.3) is transformed to the form (2.1.1), then

$$\hat{k}(u) = \Gamma(1-iu)\ \zeta(-iu)\ (1-2^{1+iu}) ,$$

where ζ is Riemann's ζ-function and Γ is the usual Γ-function. The function $\hat{k}(u+iv)$ is analytic in $w = u+iv$ for $v > 0$, but has a great number of zeros there. The following example shows that the conclusion (2.1.4) is not in general implied by the hypothesis (2.1.3).

<u>Example</u>. Let $x \in (0,1)$, we get

$$\sum_{-\infty}^{\infty} 2^n \frac{x^{2^n}}{1+x^{2^n}} = \lim_{N\to-\infty} \sum_{N}^{\infty} 2^n \left(\frac{x^{2^n}}{1-x^{2^n}} - 2\frac{x^{2^{n+1}}}{1-x^{2^{n+1}}} \right) = \tag{2.1.5}$$

$$= \lim_{N\to-\infty} 2^N \frac{x^{2^N}}{1-x^{2^N}} = -\frac{1}{\log x} .$$

If x is replaced by $x^{\sqrt{2}}$ in (2.1.5) , then

$$\sum_{-\infty}^{\infty} (\sqrt{2})^{2n+1} \frac{x^{(\sqrt{2})^{2n+1}}}{1+x^{(\sqrt{2})^{2n+1}}} = -\frac{1}{\log x} . \tag{2.1.6}$$

From (2.1.5) and (2.1.6) it follows that

$$\sum_{-\infty}^{\infty} (-\sqrt{2})^n \frac{x^{\sqrt{2}^n}}{1+x^{\sqrt{2}^n}} = 0 .$$

Hence,

$$\sum_{0}^{\infty} (-\sqrt{2})^n \frac{x^{\sqrt{2}^n}}{1+x^{\sqrt{2}^n}} = -\sum_{1}^{\infty} (-\sqrt{2})^{-n} \frac{x^{\sqrt{2}^{-n}}}{1+x^{\sqrt{2}^{-n}}} .$$

The series to the right has a limit when $x \uparrow 1$, and therefore this also holds for the series to the left, but

$$\sum_{0}^{\infty} (-\sqrt{2})^n \text{ is not convergent.}$$

The possibility of this example is the zeros of $\hat{k}(u+iv)$ as will be seen later.

2.2. A Tauberian theorem.

To obtain a high indices theorem for kernels, where the extension of the Fourier transform has zeros in the upper halfplane, we may have to invoke restrictions on the sequence $\{\lambda_n\}$ beyond being well separated as in (2.1.2).

The main problem of this section is to see how the zeros of $\hat{k}(u+iv)$ in $v \geq 0$ interact with the sequence $\{\lambda_n\}$.

If

$$a_n = O(\exp(A\lambda_n)) \ , \ n \to \infty \ ,$$

where the sequences $\{a_n\}$ and $\{\lambda_n\}$ are those occuring in (2.1.1) and the latter satisfies (2.1.2), then

$$h(w) = \sum_{1}^{\infty} a_n \, e^{iw\lambda_n}$$

is well defined as a Dirichlet series for Im $w > A$. The function h and its connection with F in (2.1.1) will be important throughout this section.

We need a definition before stating the main theorem.

Definition. *Let* $\{\mu_n\}$ *be a sequence of positive numbers satisfying*

$$\liminf_{n \to \infty} (\mu_{n+1} - \mu_n) = c > 0 \ .$$

Let

$$N(r) = \#\{\mu_n : \mu_n \leq r\}.$$

Then

$$D = \lim_{\xi \uparrow 1} \limsup_{r\to\infty} \frac{N(r) - N(r\xi)}{r - r\xi}$$

is called the maximal density of the sequence $\{\mu_n\}$.

Remark. $D \leq 1/c$.

The definition is due to Pólya ([14] p. 559).

Let I be a subset of $\mathbb{R}$. We denote its Lebesgue measure by $|I|$.

All sums are taken from 1 to ∞.

Theorem 2.1. Suppose that the coefficients $\{a_n\}$ satisfy

$$a_n = O(\exp(A\lambda_n)),$$

where A is a constant and $\{\lambda_n\}$ is a sequence of positive numbers with maximal density D and such that

$$\lambda_{n+1} - \lambda_n \geq c > 0.$$

Let the kernel $k \in \mathcal{S}$ and suppose that its Fourier transform $\hat{k}$, with $\hat{k}(0) \neq 0$, can be analytically continued into the strip $0 < \operatorname{Im} w < v_o$, $v_o > A$, and that

$$\hat{k}(\cdot + iv) \in L^2(\mathbb{R}), \quad 0 < v < v_o.$$

Moreover, suppose that on every line $\operatorname{Im} w = \sigma$, $0 \leq \sigma \leq A$, there exists a closed interval I such that

$$\hat{k}(u+i\sigma) \neq 0 \quad \text{for} \quad u \in I,$$

where $|I| = 2\pi D$ if $0 < \sigma \leq A$ and $|I| = 2\pi/c$ if $\sigma = 0$.

Let

$$F(x) = \sum a_n \int_{\lambda_n - x}^{\infty} k(y)\,dy, \quad x \in \mathbb{R}, \tag{2.2.1}$$

and suppose that

$$\lim_{x\to\infty} F(x) = 0. \tag{2.2.2}$$

Then

$$\Sigma\, a_n = 0. \tag{2.2.3}$$

In order to prove theorem 2.1, we need several auxiliary results, but first two more notations:

G is used for the function defined by

$$G(x) = \begin{cases} 0, & x > 0 \\ 1, & x < 0 \end{cases}$$

and f_v denotes the function, whose value at $x \in \mathbb{R}$ is

$$f_v(x) = f(x)e^{vx}.$$

Lemma 2.2. Suppose that there exists a constant A such that

$$a_n = O\,(\exp(A\,\lambda_n)), \tag{2.2.4}$$

where the positive numbers $\{\lambda_n\}$ satisfy

$$\lambda_{n+1} - \lambda_n \geq c > 0,\ n = 1,2,\ldots\ . \tag{2.2.5}$$

Suppose that

$$k_{v_0} \in L^2(\mathbb{R}) \quad \text{for some} \quad v_0 > A. \tag{2.2.6}$$

Let

$$F(x) = \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy. \tag{2.2.7}$$

Then there exists a function g such that

$$F(x) = \int_{-x}^{\infty} g(t)dt \tag{2.2.8}$$

and

$$\hat{g}_{v_0} = \hat{k}(\cdot + iv_0)\ h(\cdot + iv_0) \in L^2(\mathbb{R}). \tag{2.2.9}$$

Proof. We rewrite (2.2.7) as

$$(2.2.10)\qquad F(x) = \Sigma a_n \int k(y)\, G(\lambda_n - x - y)dy =$$

$$= e^{v_o x} \Sigma a_n e^{-v_o \lambda_n} \int k_{v_o}(y) G_{v_o}(\lambda_n - x - y)dy =$$

$$= e^{v_o x} \Sigma a_n e^{-v_o \lambda_n} \int \hat{k}_{v_o}(u) \hat{G}_{v_o}(u) e^{iu(\lambda_n - x)} du,$$

where the last equality is Parseval's formula.

By Fubini's theorem, the order of summation and integration can be changed in (2.2.10), and we find that

$$(2.2.11)\qquad F(x) = e^{v_o x} \int \hat{k}(u+iv_o)\hat{G}(u+iv_o)\left(\Sigma a_n e^{i(u+iv_o)\lambda_n}\right) e^{-iux}\, du$$

$$= \int \hat{k}(u+iv_o)\hat{G}(u+iv_o)\; h(u+iv_o)\; e^{-i(u+iv_o)x}\, du.$$

Put

$$\hat{g}(u+iv_o) = \hat{k}(u+iv_o)\; h(u+iv_o).$$

From (2.2.4) and (2.2.5) it follows that $h(\cdot+iv_o) \in L^\infty(\mathbb{R})$. This together with (2.2.6) gives $\hat{g}(\cdot+iv_o) \in L^2(\mathbb{R})$. Thus, there exists a function g such that

$$(2.2.12)\qquad (g(y)e^{v_o y})\hat{\ }(u) = \hat{g}(u+iv_o).$$

Insertion of (2.2.12) in (2.2.11) gives

$$F(x) = \int (g(y)e^{v_o y})\hat{\ }(u)\; \hat{G}(u+iv_o)\; e^{-i(u+iv_o)x} =$$

$$= e^{v_o x} \int g(y)e^{v_o y}\; G_{v_o}(-x-y)dy = \int_{-x}^{\infty} g(y)dy$$

and (2.2.8) is proved.

Next we need a result about tempered distributions formulated as

Lemma 2.3. Let $\Theta \in \mathcal{S}'$ and suppose that $\operatorname{supp} \Theta \subset (-\infty,0)$. Then the Fourier transform $\hat{\Theta}$ has an analytic extension into the upper halfplane.

Proof. Let ϕ be a C^∞-function such that $\phi = 1$ on $\operatorname{supp} \Theta$ and $\operatorname{supp} \phi \subset (-\infty,0]$. If $v > 0$, then

$$(e^{vx}\Theta)\hat{\ } = (e^{vx}\,\phi\,\Theta)\hat{\ } = \hat{\Theta} * \hat{\phi}_v,$$

since $\phi_v \in \mathcal{S}$, and we get

$$(e^{vx}\Theta)\hat{}(u) = \hat{\Theta} * \hat{\phi}_v(u) = \hat{\Theta}(\tau_u(\hat{\phi}_v)^{\vee}) = \hat{\Theta}(\hat{\phi}(u+iv-\cdot)) = \hat{\Theta}(u+iv).$$

To prove that $\hat{\Theta}(u+iv)$ is analytic in $w = u+iv$, $v > 0$, we use Morera's theorem. Let Γ be a regular closed curve in the upper halfplane. The map

$$\Gamma \ni w \longrightarrow \hat{\phi}(w-\cdot) \in \mathcal{S}.$$

is continuous, because

$$D_t^k [\hat{\phi}(w-t) - \hat{\phi}(w_o-t)] = \int (ix)^k \phi(x) (e^{-iwx} - e^{-iw_ox}) e^{itx} dx$$

$$= \frac{(-1)^N}{(it)^N} \int D_x^N [(ix)^k \phi(x) (e^{-iwx} - e^{-iw_ox})] e^{itx} dx,$$

where the second equality is given after integrations by parts N times.

Hence

$$|t|^N |D_t^k(\hat{\phi}(w-t) - \hat{\phi}(w_o-t))| = \left|\int D_x^N[x^k\phi(x) (e^{-iwx}-e^{-iw_ox})]e^{itx}dx\right|$$

and the right side of this equality tends to zero when $w \to w_o$, by dominated convergence.

Thus, we have that

(2.2.13) $$H = \int_\Gamma \hat{\phi}(w-\cdot)\, dw$$

is a well defined $\mathcal{S}$-valued integral (see for instance Rudin [16], chapter 3). Evaluation, at any $t \in \mathbb{R}$, is a tempered distribution, and therefore we find that

(2.2.14) $$H(t) = \int_\Gamma \hat{\phi}(w-t)\, dw = 0 ,$$

because $\hat{\phi}(w)$ is analytic for $\operatorname{Im} w > 0$. Apply $\hat{\Theta}$ in (2.2.13) and we get, with the use of (2.2.14), that

$$0 = \int_\Gamma \hat{\Theta}(\hat{\phi}(w-\cdot))\, dw = \int_\Gamma \hat{\Theta}(w)\, dw ,$$

i.e. $\hat{\Theta}(w)$ is analytic in the upper halfplane.

The proof will be completed by proving

(2.2.15) $\hat{\Theta}(\cdot+iv) \to \hat{\Theta}$ in $\mathcal{S}'$, $v \downarrow 0$.

It is enough to prove that

(2.2.16) $\psi_v \to \psi$ in $\mathcal{S}$, $v\downarrow 0$,

where supp $\psi \subset [-\infty,0]$, because if (2.2.16) is true, then

$$\lim_{v\downarrow 0} \hat{\Theta}_v((\hat{\psi})^\vee) = \lim_{v\downarrow 0} \Theta_v(\psi) = \lim_{v\downarrow 0} \Theta(\psi_v) = \Theta(\psi) = \hat{\Theta}((\hat{\psi})^\vee)$$

and (2.2.15) is proved. The convergence in (2.2.16) is easily obtainable from the definition, because

$$|x|^N |D^k(\psi_v-\psi)(x)| \le |x|^N \{ |(e^{vx}-1)\, D^k\psi(x)| + vC_k \sum_{j=1}^{k-1} |D^j\psi(x)|\} \le$$

$$\le v\, \{|x|^{N+1}|D^k\psi(x)| + C_k \sum_{j=1}^{k-1} |x|^N|D^j\psi(x)|\} \to 0 , \; v\downarrow 0 .$$

Putting together lemma 2.2 and 2.3, we obtain

<u>Lemma 2.4</u>. <u>With the same assumptions and notation as in lemma 2.2 and furthermore that</u> $F \in \mathcal{S}'$, <u>it holds that</u> $g \in \mathcal{S}'$ <u>and that its Fourier transform</u> $\hat{g}$ <u>has an analytic extension into the strip</u> $0 < \operatorname{Im} w < v_0$.

<u>Proof</u>. From (2.2.8) we have that

$$F(x) = \int_{-x}^{\infty} g(t)\, dt \quad , \quad x \in \mathbb{R} .$$

Since $F \in \mathcal{S}'$, we get that $g \in \mathcal{S}'$. Thus g has Fourier transform as a tempered distribution.

Let ϕ_1 be a C^{∞}-function with supp $\phi_1 \subset (-\infty, 0)$ and $\phi_1(x) = 1$ if $x < -1$. Take $\phi_2 = 1 - \phi_1$. From lemma 2.3 we get that

(2.2.17) $\quad (\phi_1 g)\hat{}\,(w)$ is analytic for $\operatorname{Im} w > 0$.

Moreover, (2.2.9) states that

$$\int |g(t)\, e^{v_0 t}|^2\, dt < \infty .$$

Hence,

(2.2.18) $$\int |\phi_2(t)\, g(t)\, e^{v_0 t}|^2\, dt < \infty .$$

The convergence in (2.2.18) can obviously be extended to hold for $v < v_0$ and this implies that

(2.2.19) $\quad (\phi_2 g)\hat{}\,(w)$ is analytic for $\operatorname{Im} w < v_0$.

Putting (2.2.17) and (2.2.19) together, we get that

$$\hat{g}(w) = ((\phi_1 + \phi_2)g)\hat{}\,(w) = (\phi_1 g)\hat{}\,(w) + (\phi_2 g)\hat{}\,(w)$$

is analytic in the strip $0 < \operatorname{Im} w < v_0$.

<u>Remark.</u> By lemma 2.4 we can now see the explanation of the example in the introduction of this chapter. In that case the function h is

$$h(w) = \sum_{0}^{\infty} (-\sqrt{2})^n\, e^{iwn \log \sqrt{2}} = (1 + e^{(1+iw)\log \sqrt{2}})^{-1} \text{ for } \operatorname{Im} w > 1 ,$$

which has an analytic extension except for simple poles on $\operatorname{Im} w = 1$. Since

$$\hat{k}(w) = \Gamma(1-iw)\, \zeta(-iw)\, (1-2^{1+iw}) ,$$

we have that

$$\hat{g}(w) = \hat{k}(w) \cdot h(w) = \Gamma(1-iw)\, \zeta(-iw)\, (1 - e^{(1+iw)\log \sqrt{2}}) ,$$

and this product is analytic for $\operatorname{Im} w > 0$.

In order to come further we need lemma 1.2. This interpolation result will be used to get global information from local behavior of a Dirichlet series on the axis of convergence. More precisely

<u>Theorem 2.5</u>. <u>Let</u> f <u>be defined by a Dirichlet series,</u>

$$f(w) = \Sigma \alpha_n e^{iw\lambda_n} \quad , \quad \operatorname{Im} w > 0 \ .$$

<u>Let</u> $\{\lambda_n\}$ <u>be a sequence of positive numbers such that</u>

$$\lambda_{n+1} - \lambda_n \geq c > 0 \ .$$

<u>Suppose that</u>

$$f(\cdot + iv) \to \tilde{f} \quad \text{in } \mathcal{D}'(I) \ , \quad v \downarrow 0 \ , \tag{2.2.20}$$

<u>for some open interval</u> I <u>with</u> $|I| > 2\pi/c$.

<u>Then there exist a finite constant</u> N <u>and a discrete measure</u> $\mu \in \mathcal{S}'$ <u>of the form</u>

$$\mu = 2\pi \ \Sigma \ \alpha_n \ \tau_{-\lambda_n} \delta \ , \qquad \alpha_n = O(\lambda_n^N),$$

<u>such that</u>

$$\hat{\mu} = \tilde{f} \quad \text{in } \mathcal{S}' \quad .$$

(δ is the Dirac measure).

<u>Proof</u>. There is no loss in generality to assume that the interval I is symmetric around the origin. Then we can choose an $\varepsilon > 0$ such that the functions $\{H_k\}$, from lemma 1.2, have $\operatorname{supp}(\hat{H}_k) \subset [-\frac{\pi}{c} - \varepsilon, \frac{\pi}{c} + \varepsilon] \subset I$.

Since $\tilde{f}$ is a bounded functional on $\mathcal{D}(I)$, we have

$$|\tilde{f}(\hat{H}_k)| \leq C \ \|H_k\|_N \quad , \quad k = 1,2 \ldots . \tag{2.2.21}$$

for some finite N and some constant C.

Apply (2.2.20) to $\hat{H}_k$ and we get

$$(2.2.22) \qquad \tilde{f}(\hat{H}_k) = \lim_{v\downarrow 0} f(\cdot+iv)\,(\hat{H}_k) = \lim_{v\downarrow 0} \int f(u+iv)\,\hat{H}_k(u)\,du =$$

$$= \lim_{v\downarrow 0} \Sigma\, \alpha_n\, e^{-v\lambda_n} \int \hat{H}_k(u)\, e^{iu\lambda_n}\,du =$$

$$= \lim_{v\downarrow 0} \Sigma\, \alpha_n\, e^{-v\lambda_n}\, 2\pi\, H_k(\lambda_n) = 2\pi\alpha_k \ .$$

If (2.2.21) and (2.2.22) are combined with (1.2.5) and (1.2.6), we see that

$$|2\pi\alpha_k| \le C\, \|\hat{H}_k\|_N = O(\lambda_k^N) \ , \quad k\to\infty \ ,$$

and the theorem is proved.

Corollary 2.6. If the assumptions in theorem 2.5 are fulfilled except that (2.2.20) is replaced by

$$(2.2.23) \qquad f(\cdot+iv) \to \tilde{f} \text{ in } L^1(I) \ , \quad v\downarrow 0 \ ,$$

then

$$(2.2.24) \qquad |\alpha_n| \le C\, \|\tilde{f}\|_{L^1(I)} \ ,$$

where C is a constant depending only on c and $|I|$.

Corollary 2.6 easily implies a weaker form of Pólya's theorem. (See Levinson [12], theorem XXIX; Pólya [13]). This result is stated in the next corollary.

Corollary 2.7. Let

$$f(w) = \Sigma\, \alpha_n\, e^{iw\lambda_n}$$

be a Dirichlet series with the axis of convergence $\text{Im } w = \sigma$. Suppose that the positive numbers $\{\lambda_n\}$ satisfy

$$(2.2.25) \qquad \liminf_{n\to\infty} (\lambda_{n+1}-\lambda_n) = c > 0 \ .$$

Then every closed interval of the length $2\pi/c$ on the line $\operatorname{Im} w = \sigma$ contains at least one singular point of f.

Proof. Suppose that there exists a closed interval I, $|I| = 2\pi/c$, on the axis of convergence, where f has not any singular point. It is no loss in generality to assume that $\sigma = 0$ and $I = [-\pi/c, \pi/c]$. Since the points of singularity form a closed set, it exists an $\varepsilon > 0$ such that the function $f(w)$ can be analytically continued into the domain

$$\{w \in \mathbb{C};\ \operatorname{Im} w > -2\varepsilon,\quad -\pi/c - 2\varepsilon < \operatorname{Re} w < \pi/c + 2\varepsilon\}.$$

From (2.2.25) it follows that there exists a positive integer N_0 such that

$$\lambda_{n+1} - \lambda_n \geq \frac{c}{1+\varepsilon \cdot c/4\pi} = c_1, \quad n \geq N_0.$$

If we define $f_1(w) = \sum_{N_0}^{\infty} \alpha_n e^{iw\lambda_n}$, $\operatorname{Im} w > 0$, then this function can be analytically extended into the same domain as $f(w)$.

Let $I' = [-\pi/c - \varepsilon/2, \pi/c + \varepsilon/2]$. By Cauchy's estimate we get

$$|f_1^{(k)}(u)| \leq \frac{k!\, M}{\varepsilon^k}, \quad u \in I'. \tag{2.2.26}$$

For $\operatorname{Im} w > 0$, the k'th derivative of f_1 is

$$f_1^{(k)}(w) = \sum_{N_0}^{\infty} \alpha_n (i\lambda_n)^k e^{iw\lambda_n}.$$

Since $|I'| > 2\pi/c_1$ and the hypothesis (2.2.23) obviously is fulfilled for each derivative of f_1, we get as in (2.2.24) that

$$|\alpha_n|\, \lambda_n^k \leq C \int_{I'} |f_1^{(k)}(u)|\, du. \tag{2.2.27}$$

Inserting (2.2.26) in (2.2.27), we find that

$$|\alpha_n| \frac{\lambda_n^k}{k!} (\varepsilon/2)^k \leq CM |I'| \cdot 2^{-k}, \quad k = 0,1,2, \ldots .$$

Hence,

$$\alpha_n = O(\exp(-\lambda_n \varepsilon/2)), \; n \to \infty .$$

But this implies that the axis of convergence $\operatorname{Im} w = \sigma \leq -\varepsilon/2$, which contradicts the hypothesis and proves the corollary.

Remark. Pólya's theorem says: on the axis of convergence the Dirichlet series has at least one singular point in every closed interval of length $2\pi D$. (D is the maximal density for the sequence $\{\lambda_n\}$).

Proof of theorem 2.1. From lemma 2.2 and 2.4 we have

$$F(x) = \int_{-x}^{\infty} g(t)\, dt$$

and

(2.2.28) $\hat{g}(w) = \hat{k}(w)\, h(w)$ is analytic for $0 < \operatorname{Im} w < v_0$.

Suppose that the axis of convergence for the Dirichlet series $h(w)$ is $\operatorname{Im} w = \sigma > 0$. Then, on the line $\operatorname{Im} w = \sigma$ the Dirichlet series $h(w)$ has at least one singular point in each closed interval of length $2\pi D$. The formula (2.2.28) shows that $h(w)$ has a meromorphic extension into the domain $\operatorname{Im} w > 0$, and therefore these singularities must be poles. Moreover, the product $\hat{k}(w) \cdot h(w)$ is analytic for $\operatorname{Im} w > 0$, but this contradicts the hypotheses on the $\hat{k}(w)$. Thus, the series defining h converges not only for $\operatorname{Im} w > A$ but

(2.2.29) $h(w) = \Sigma a_n e^{iw\lambda_n}$, $\operatorname{Im} w > 0$.

The assumptions on the Fourier transform $\hat{k}$ imply that there exist a closed interval I, with $|I| > 2\pi/c$, and an $\varepsilon > 0$ such that

(2.2.30) $\hat{k}(u+iv) \neq 0$, when $u \in I$, $0 \leq v < \varepsilon$,

and

(2.2.31) $D^n \hat{k}(\cdot+iv) \to D^n \hat{k}$ in uniform norm on I, $v \downarrow 0$, $n=0,1,\dots$.

From the proof of lemma 2.4, it follows that

(2.2.32) $\hat{g}(\cdot+iv) \to \hat{g}$ in $\mathscr{S}'$, $v \downarrow 0$.

Combining (2.2.30), (2.2.31) and (2.2.32), we have (see Rudin [16], theorem 6.18) that

(2.2.33) $h(\cdot+iv) \to \tilde{h}$ in $\mathscr{D}'(\text{Int } I)$, $v \downarrow 0$.

Since (2.2.29) and (2.2.33) hold, we can use theorem 2.5 to get

(2.2.34) $a_n = O(\lambda_n^N)$, $n \to \infty$.

Recalling (2.2.1) and (2.2.2), we see that

$$\lim_{x\to\infty} \sum a_n \int_{\lambda_n - x}^{\infty} k(y)dy = 0 ,$$

where the coefficients $\{a_n\}$ satisfy (2.2.34). Now we can use theorem 1.3 to get the conclusion (2.2.3), and theorem 2.1 is proved.

If $\lambda_n = nc$, $n = 1,2,\dots$, then we can say more than theorem 2.1. We state the following corollary.

Corollary 2.8. *Let the coefficients* $\{a_n\}$ *satisfy*

$$a_n = O(e^{A\lambda_n}), \quad n \to \infty,$$

where $\lambda_n = nc$, $n = 1,2,\dots$ *and* $c > 0$.

Suppose that the kernel $k \in \mathscr{S}$ *has a Fourier transform* $\hat{k}$, *which can be analytically extended into the strip* $0 < \text{Im } w < v_0$, $v_0 > A$, *with*

$$\hat{k}(\cdot+iv) \in L^2(R) , \quad 0 < v < v_0 .$$

Suppose further that $\hat{k}(0) \neq 0$. Let

$$F(x) = \Sigma\, a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy\ .$$

Then a sufficient condition that

(2.2.35) $$\lim_{x\to\infty} F(x) = 0 \quad \text{implies} \quad \Sigma\, a_n = 0$$

is that on no line $\operatorname{Im} w = \sigma$, $0 \leq \sigma \leq A$, $\hat{k}$ has zeros separated exactly by the distance $2\pi m/c$, $m = 1,2, \ldots$.

Moreover, if there exists a line $\operatorname{Im} w = \sigma$, $0 \leq \sigma \leq A$, where $\hat{k}(w)$ has zeros separated exactly by the distance $2\pi m/c$, $m = 1,2, \ldots$, then the coefficients $\{a_n\}$ can be chosen in such a way that (2.2.35) is false.

Proof. The same arguments as in the proof of theorem 2.1 are used to prove the first part of the corollary. To prove the second part, we note that there is an $\alpha \geq 0$ such that

$$\hat{k}(\alpha + i\sigma + 2\pi m/c) = 0\ ,\ m \in \mathbf{Z}\ .$$

Let

$$a_n = \exp(c(\sigma - i\alpha)n)\ ,\ n = 0,1, \ldots$$

and define

$$h(w) = \sum_0^{\infty} a_n\, e^{iwcn} = (1 - \exp\,(c(\sigma - i\alpha + iw)))^{-1}\ ,\ \operatorname{Im} w > \sigma\ .$$

Thus,

$$\hat{g}(w) = \hat{k}(w)\, h(w) \quad \text{is analytic for} \quad \operatorname{Im} w > 0.$$

and

$$\lim_{x\to\infty} F(x) = \int g(t)\, dt = \hat{k}(0)\ (1 - \exp(c(\sigma - i\alpha)))^{-1}.$$

but

$$a_n \not\to 0\ ,\ n \to \infty.$$

Remark. Suppose that the Fourier transform $\hat{k}(w)$ has at least one zero in every closed interval of length $2\pi D$ on the line $\operatorname{Im} w = \sigma > 0$, but on no other line. If we have divergence of the sum of the coefficients $\{a_n\}$, then the coefficients are restricted in such a way that the Dirichlet series $h(w)$ has $\operatorname{Im} w = \sigma$ as the axis of convergence. Moreover, $h(w)$ has poles on $\operatorname{Im} w = \sigma$.

If $\hat{k}(w)(1-\exp(c(\sigma-i\alpha+iw)))^{-1} = \hat{k}_1(w)$ and $\hat{k}_1$ satisfies the hypotheses in corollary 2.8, then the divergence of Σa_n implies that

$$\Sigma(a_n - e^{(\sigma-i\alpha)c}\, a_{n-1}) \text{ is convergent.}$$

In particular, if $\sin\alpha c \neq 0$ and the coefficients $\{a_n\}$ are real, then Σa_n must converge.

2.3. Estimation of the coefficients.

The main result in this section is an estimate of the coefficients $\{a_n\}$, when the growth of the function $F(x)$ in (2.2.1) is known. The theorem is very close to Levinson's theorem B, but does not follow from that result. The proof given below of theorem 2.9 is a slight modification of the proof of theorem B.

We give a definition and use the previous notation k_α for the function defined by $k_\alpha(x) = k(x)\, e^{\alpha x}$.

Definition. Suppose that k is a function such that $k_\alpha \in L^1(\mathbb{R})$, and that its Fourier transform $\hat{k}_\alpha$ has an extension, which is analytic in the open and continuous in the closed upper halfplane, and that $\hat{k}_\alpha'(u)$ exists, $u \in \mathbb{R}$. Let $\beta > 0$ and ξ be real, and suppose that

$$(2.3.1) \qquad \max_{|\xi| \le \beta} \left| \frac{\hat{k}_\alpha(u+\xi)}{\hat{k}_\alpha(u)} \right| \le e^{\Theta(u)}$$

$$(2.3.2) \qquad \max_{|\xi|\leq\beta} \left| \frac{\hat{k}_\alpha(w+\xi)}{\hat{k}_\alpha(w)} \right| \leq Ce^{C|w|}, \text{ Im } w \geq 0, \text{ (C is a constant)}$$

$$(2.3.3) \qquad \left| \frac{\hat{k}'_\alpha(u)}{\hat{k}_\alpha(u)} \right| \leq e^{\Theta(u)},$$

where $\Theta(u)$ *is a positive even function of* u, *monotonely increasing for* $u > 0$, *and*

$$\int_1^\infty \frac{\Theta(u)}{u^2}\, du < \infty .$$

Then we say that $k \in \mathcal{L}_\alpha$.

Theorem 2.9. Suppose that there exists an $\alpha \geq 0$ *such that*

$$(2.3.4) \qquad e^{-\alpha x} \sum a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy$$

converges uniformly on every set $\{x : x < x_0\}$, $x_0 < \infty$, *and defines a bounded function on* $\mathbb{R}$.

The positive numbers $\{\lambda_n\}$ *satisfy*

$$\lambda_{n+1} - \lambda_n \geq c > 0, \quad n = 1,2, \ldots ,$$

and $k \in \mathcal{L}_\alpha$. *Then*

$$(2.3.5) \qquad a_n = O(e^{\alpha\lambda_n}) .$$

In order to prove this theorem we need two auxiliary results. The first is almost identical with lemma 34.2 in [12].

Lemma 2.10. There exists a function ψ, analytic in the upper halfplane, with

(2.3.6) $$|\psi(u+iv)| \leq C\, e^{Cv}, \quad (C \text{ is a constant})$$

$|\psi(u)|$ is even

and

(2.3.7) $$|\psi(u)| \leq C_N(1+|u|^N)^{-1} \exp(-\Theta(u)), \quad N = 1,2, \dots ,$$

where $\{C_N\}$ are finite constants and $\Theta(u)$ is a positive even function of u, monotonely increasing for $u > 0$, and with

$$\int_1^\infty \frac{\Theta(u)}{u^2} < \infty .$$

Moreover,

(2.3.8) $$\hat{\psi}(0) = 1.$$

Proof. Let $\Theta_1(t) = 2\Theta(t) + 2\,|t|^{1/2}$ and define

$$\gamma(u,v) = \frac{1}{\pi} \int \frac{(v+1)\,\Theta_1(t)}{(u-t)^2 + (v+1)^2}\, dt .$$

For $u > 0$,

(2.3.9) $$\gamma(u,0) \geq \frac{1}{\pi} \int_u^\infty \frac{\Theta_1(t)}{(u-t)^2+1}\, dt \geq \frac{\Theta_1(u)}{\pi} \int_0^\infty \frac{dt}{t^2+1} = \Theta(u) + |u|^{1/2} .$$

Since $\gamma(u,0)$ is even, the inequality (2.3.9) holds for all $u \in \mathbb{R}$. Let $\tilde{\gamma}(u,v)$ be the harmonic conjugate of $\gamma(u,v)$ in $v > -1$. Then

(2.3.10) $$\psi_1(u+iv) = \exp\{-\gamma(u,v) - i\,\tilde{\gamma}(u,v)\}$$

is analytic in $w = u+iv$ for $v > -1$. Since $\gamma(u,v) \geq 0$, we have

(2.3.11) $$|\psi_1(u+iv)| \leq 1.$$

From (2.3.9) and (2.3.10) it follows that

(2.3.12) $\qquad |\psi_1(u)| \leq \exp\,(\,-\,\Theta(u)\,-\,|u|^{1/2})\,.$

The Fourier transform $\hat{\psi}_1$ is not identically zero. Thus there exist numbers a and b such that $\hat{\psi}_1(a) = b \neq 0$. Define

$$\psi(u) = \frac{1}{b}\,\psi_1(u)\,e^{-iua}\,.$$

This in combination with (2.3.11) and (2.3.12) proves the lemma.

The next lemma is the crucial part in the proof of theorem 2.9.

<u>Lemma 2.11</u>. <u>Suppose the hypotheses in theorem 2.9 are satisfied. Then there exists a sequence</u> $\{B_n\}_1^\infty$ <u>of functions, defined on</u> $R \times [2,\infty)$, <u>such that</u>

(2.3.13) $\qquad \int |B_n(x,A)|\,dx \leq C_1\,,$

(2.3.14) $\qquad B_n(x,A) = 0 \quad \text{for} \quad x \geq C_2 A\,,$

(2.3.15) $\qquad \int B_n(x,A)\,e^{\alpha(\lambda_m - x)}\,(\int_{\lambda_m - x}^{\infty} k(y)dy)dx = 0,\ m \neq n\,,$

<u>and</u>

(2.3.16) $\qquad \lim_{A\to\infty} \int B_n(x,A)\,e^{\alpha(\lambda_n - x)}\,(\int_{\lambda_n - x}^{\infty} k)y)dy)\,dx = \hat{k}_\alpha(0)\,,$

<u>where</u> C_1 <u>and</u> C_2 <u>are constants</u>.

<u>Proof</u>. Define

(2.3.17) $\qquad \hat{B}_n(u,A) = -\,\dfrac{i(u+i\alpha)}{\hat{k}_\alpha(u)}\,(k_\alpha\cdot\psi^A) * \hat{H}_n(u)\,,\quad n = 1,2,\,\ldots\,,$

where the functions $\{H_n\}$ are given in lemma 1.2 with $\text{supp}(\hat{H}_n) \subset (-\beta,\beta)$, $\beta > \pi/c$. The function ψ^A is defined by $\psi^A(u) = A\,\psi(Au)$, where ψ is the function in lemma 2.10, except that we instead for $\Theta(u)$ here use $2\Theta(u)$.

By (1.2.6) and (2.3.1), we get

$$|\hat{B}_n(u,A)| \le C_3 \, |u| \, e^{\Theta(u)} \int_{u-\beta}^{u+\beta} |\psi^A(t)| dt =$$

$$= C_3 |u| \, e^{\Theta(u)} \int_{A(u-\beta)}^{A(u+\beta)} |\psi(t)| dt.$$

If $|u| > 2\beta$, then from (2.3.7) it follows that

$$|\hat{B}_n(u,A)| \le C_3 \, |u| \, e^{\Theta(u)} \int_{|u|}^{\infty} |\psi(t)| \, dt \le \frac{C_4}{1 + u^2} . \tag{2.3.18}$$

For $|u| \le 2\beta$ the inequality (2.3.18) is obvious. In a similar manner, by using (1.2.6), (2.3.1), (2.3.3) and (2.3.7), we can estimate

$$\frac{d}{du} \{e^{iu\lambda_n} \hat{B}_n(u,A)\} = - \frac{i(u+i\alpha)}{\hat{k}_\alpha(u)} \int \hat{k}_\alpha(t) \psi^A(t) \, e^{it\lambda_n} \, \hat{G}_n'(u-t) \, dt$$

$$- \frac{i}{\hat{k}_\alpha(u)} \left(1 - (u+i\alpha) \frac{\hat{k}_\alpha'(u)}{\hat{k}_\alpha(u)}\right) \int \hat{k}_\alpha(t) \, \psi^A(t) \, e^{it\lambda_n} \, \hat{G}_n(u-t) \, dt ,$$

where $\hat{G}_n(t) = \hat{H}_n(t) \, e^{i\lambda_n t}$, to get

$$\left| \frac{d}{du} \{e^{iu\lambda_n} \hat{B}_n(u,A)\} \right| \le \frac{C_5}{1+u^2} .$$

Thus, by Carlson's inequality

$$\int |B_n(x,A)| \, dx \le C_6 .$$

We have that

$$\hat{B}_n(w,A) = - \frac{i(w+i\alpha)}{\hat{k}_\alpha(w)} \int \hat{k}_\alpha(w-t) \, \psi^A(w-t) \, \hat{H}_n(t) \, dt$$

is analytic for $\operatorname{Im} w > 0$ and continuous for $\operatorname{Im} w \ge 0$. Using (1.2.6), (2.3.2) and (2.3.6), we find that

$$|\hat{B}_n(w,A)| \leq C_7 \; |w| \; A \; e^{C_8 A v} \; e^{C_9 |w|} \leq C_7 \; e^{C_{10} A |w|} . \tag{2.3.19}$$

By a Phragmén-Lindelöf argument, (2.3.18) and (2.3.19) imply that

$$|\hat{B}_n(w,A) \; e^{iC_{10} A w}| \leq \frac{C_{11}}{1+|w|^2} , \quad \text{Im } w \geq 0 ,$$

Hence, applying Paley-Wiener's theorem, we get

$$B_n(x,A) = 0 \quad \text{for} \quad x \geq C_{10} A .$$

Parseval's formula gives

$$\frac{1}{2\pi} \int \hat{B}_n(u,A) \; \hat{k}_\alpha(u) \; e^{iut} \, du = \int B_n(x,A) \; k_\alpha(t-x) \, dx =$$

$$= \int B_n(x,A) \; e^{\alpha(t-x)} \; k(t-x) \, dx$$

and this can be rewritten as

$$\frac{1}{2\pi} \int \hat{B}_n(u,A) \; \hat{k}_\alpha(u) \; e^{i(u+i\alpha)t} \, du = \int B_n(x,A) \; e^{-\alpha x} \; k(t-x) \, dx. \tag{2.3.20}$$

Integrating (2.3.20) with respect to t, we get

$$\frac{1}{2\pi} \int \hat{B}_n(u,A) \; \hat{k}_\alpha(u) \; \frac{e^{i(u+i\alpha)T} - e^{i(u+i\alpha)s}}{i(u+i\alpha)} \, du = \tag{2.3.21}$$

$$= \int B_n(x,A) \; e^{-\alpha x} \int_s^T k(t-x) \, dt .$$

We have

$$\lim_{T\to\infty} \int \frac{\hat{B}_n(u,A) \; \hat{k}_\alpha(u)}{i(u+i\alpha)} \; e^{i(u+i\alpha)T} \, du = 0 ,$$

which is obvious if $\alpha > 0$ and follows from Riemann-Lebesque's lemma for $\alpha = 0$. Thus, if $T \to \infty$ in (2.3.21), then

$$\int B_n(x,A) \; e^{\alpha(s-x)} \int_{s-x}^{\infty} k(t) dt \, dx = - \frac{1}{2\pi} \int \frac{\hat{B}_n(u,A) \; \hat{k}_\alpha(u)}{i(u+i\alpha)} \; e^{ius} \, du .$$

Together with (2.3.17) this implies that

$$\int B_n(x,A)e^{\alpha(s-x)} \int_{s-x}^{\infty} k(t)dt\ dx = \frac{1}{2\pi} [(\hat{k}_\alpha \cdot \psi^A) * \hat{H}_n]\hat{\ }(-s) =$$

$$= H_n(s)\ (\hat{k}_\alpha \cdot \psi^A)\hat{\ }(-s).$$

Hence,

$$\int B_n(x,A)\ e^{\alpha(\lambda_m - x)} \int_{\lambda_m - x}^{\infty} k(t)dt\ dx = 0 \quad \text{if} \quad m \neq n,$$

and

$$\lim_{A\to\infty} \int B_n(x,A)\ e^{\alpha(\lambda_n - x)} \int_{\lambda_n - x} k(t)dt\ dx = \lim_{A\to\infty} (\hat{k}_\alpha \cdot \psi^A)\hat{\ }(-\lambda_n) =$$

$$= \lim_{A\to\infty} \int k_\alpha(t)\ \hat{\psi}((t-\lambda_n)/A)\ dt = \hat{k}_\alpha(0),$$

where the last equality is given by dominated convergence and (2.3.8).

Proof of theorem 2.9. From (2.3.4) and (2.3.13) it follows that

$$\int B_n(x,A)\ e^{-\alpha x}(\ \Sigma a_m \int_{\lambda_m - x} k(y)dy)\ dx = O(1). \tag{2.3.22}$$

Because of (2.3.14) and that the series converges uniformly on every set $\{x:\ x < x_0\}$, the order of integration and summation in (2.3.22) can be changed. Thus,

$$\Sigma\ a_m e^{-\alpha\lambda_m} \int B_n(x,A)\ e^{\alpha(\lambda_m - x)}\ (\int_{\lambda_m - x}^{\infty} k(y)dy)\ dx =$$

$$= \int B_n(x,A)\ e^{-\alpha x}(\Sigma\ a_m \int_{\lambda_m - x} k(y)dy)dx.$$

By the properties (2.3.15) and (2.3.16), we find that

$$\lim_{A\to\infty} \int B_n(x,A)\ e^{-\alpha x}(\Sigma\ a_m \int_{\lambda_m - x}^{\infty} k(y)dy)dx = a_n e^{-\alpha\lambda_n}\ \hat{k}_\alpha(0). \tag{2.3.23}$$

Combining (2.3.22) and (2.3.23), we get (2.3.5) and the theorem is proved.

2.4. General high indices theorem.

We can now state the main theorem of this chapter. By a combination of the

heorems 2.1 and 2.9 we have

heorem 2.12. Suppose that there exists an $\alpha \geq 0$ such that

$$\text{(2.4.1)} \qquad e^{-\alpha x} \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy$$

onverges uniformly on every set $\{x: x < x_0\}$, $x_0 < \infty$, and defines a bounded unction on $\mathbb{R}$.

et $\{\lambda_n\}$ be a sequence of positive numbers, with maximal density D, satisying

$$\lambda_{n+1} - \lambda_n \geq c > 0.$$

uppose further that

$k \in \mathscr{S} \cap \mathscr{L}_\alpha$,

$\hat{k}(\cdot + iv) \in L^2(\mathbb{R})$, $0 < v < v_0$, for some $v_0 > \alpha$,

on every line $\operatorname{Im} w = \sigma$, $0 \leq \sigma < \alpha$, there exists a closed interval I such that $\hat{k}(u+i\sigma) \neq 0$ for $u \in I$, where $|I| = 2\pi D$ if $0 < \sigma < \alpha$ and $|I| = 2\pi/c$ if $\sigma = 0$, and

$\hat{k}(0) \neq 0$.

en

$$\lim_{x \to \infty} \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy = 0$$

plies that

$$\Sigma a_n = 0.$$

mark 1. If $\lambda_n = nc$, $c > 0$, then more than theorem 2.12 can be said. See rollary 2.8.

mark 2. If $k \in \mathscr{L}_\alpha$, $\alpha = 0$, then theorem 2.12 reduces to theorem 2.9 which this case is theorem B'.

We shall now see that theorem 2.12 implies Rényi's conjecture mentioned in the introduction.

Suppose that

(2.4.2) $$\lim_{x \uparrow 1} \Sigma a_n \frac{x^{2^n}}{1+x^{2^n}} = s.$$

Hence, as is readily seen,

(2.4.3) $$(\log \frac{1}{x})^{\alpha} \Sigma a_n \frac{x^{2^n}}{1+x^{2^n}}, \quad \alpha \geq 0,$$

converges uniformly on every set $\{x: x < x_0\}$, $x_0 < 1$, and defines a bounded function on [0,1]. We transform (2.4.2) to the form (2.2.1) and get

$$\lim_{x \to \infty} \Sigma a_n \int_{n \log 2 - x}^{\infty} k(y)dy = s,$$

where

$$\hat{k}(u) = \Gamma(1-iu)\ \zeta(-iu)\ (1-2^{1+iu}).$$

By the same transformation (2.4.3) goes into (2.4.1). From well-known theorems about the Γ-function and the ζ-function it follows that $k \in \mathcal{L}_{\alpha}$, $\alpha > 1$, and that the other assumptions in theorem 2.12 are fulfilled. (See for instance [1] and [19]). The conclusion is that

$$\lim_{N \to \infty} \sum^{N} a_n = 2s.$$

Thus Rényi's conjecture is proved.

The previous kernel is just a special case in a class of functions, where theorem 2.12 may be applicable. These functions, which have been considered by Korevaar in a similar problem, are of the form

(2.4.4) $$f(x) = (\log \frac{1}{x})^{\beta} \sum_{1}^{\infty} \alpha_m x^m, \ x \in (0,1), \ f' \in L^1(0,1), \ \beta \geq 0,$$

with $\alpha_m = O(m^{\gamma})$ for some finite γ.

Suppose that

(2.4.5) $$\lim_{x \uparrow 1} \Sigma a_n f(x^{p_n}) = 0, \quad p_{n+1}/p_n \geq \theta > 1 .$$

If we transform (2.4.5) to the form (2.2.1), i.e.

$$\lim_{x\to\infty} \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy = 0 ,$$

then we have

$$\hat{k}(u) = -iu\,\Gamma(\beta-iu)\,\chi(\beta-iu) .$$

The function χ is, for $\mathrm{Re}\, w > \gamma+1$, defined by the Dirichlet series

$$\chi(w) = \sum_1^{\infty} \frac{\alpha_m}{m^w} ,$$

and has an analytic extension into $\mathrm{Re}\, w > \beta$.

It is easily seen that there exists an $\alpha \geq 0$ such that $k \in \mathcal{L}_\alpha$. For this α also holds that

$$(\log \frac{1}{x})^{\alpha} \Sigma a_n f(x^{p_n})$$

converges uniformly on every set $\{x: x < x_0\}$, $x_0 < 1$, and defines a bounded function on [0,1].

Thus, in the case of the series (2.4.4), the existence of a high indices theorem depends solely on the location of the zeros of χ as the discussion of theorem 2.12 shows.

The next example shows that the hypotheses (2.3.1) is essential in theorem 2.12 (cf Levinson [12] p. 187).

Example. Let $\lambda_n = n$ and $k(x) = e^{-x^2/2}(1-e^{-x/2})$. Then all the hypotheses in theorem 2.12 on the kernel k are fulfilled for $\alpha > 1/4$ except (2.3.1), but there exist coefficients

$$a_n = \frac{(-1)^n e^{n^2/2}}{n!} , \; n = 0,1,2, \ldots. ,$$

such that (2.4.1) is satisfied and

$$\lim_{k\to\infty} \sum_0^{\infty} a_n \int_{\lambda_n - x}^{\infty} k(y)dy \quad \text{exists.}$$

Proof. Since

$$e^{-e^{-y}} = \sum_0^\infty \frac{(-1)^n}{n!} e^{-ny},$$

we obtain that

$$e^{-e^{-y}} e^{-y^2/2} = \sum_0^\infty \frac{(-1)^n}{n!} e^{n^2/2} e^{-(n+y)^2/2}$$

and

$$e^{-e^{-y}} \cdot e^{-y^2/2} - e^{1/8} e^{-e^{-(y+1/2)}} \cdot e^{-(y+1/2)^2/2} =$$

$$= \sum_0^\infty \frac{(-1)^n}{n!} e^{n^2/2} (e^{-(n+y)^2/2} - e^{1/8-(n+y+1/2)^2/2}).$$

Thus,

$$\int_{-x}^\infty (e^{-e^{-y}} e^{-y^2/2} - e^{1/8} e^{-e^{-(y+1/2)}} \cdot e^{-(y+1/2)^2/2} =$$

$$= \sum_0^\infty \frac{(-1)^n}{n!} e^{n^2/2} \int_{n-x}^\infty e^{-y^2/2} (1 - e^{-y/2}) dy$$

and

$$\lim_{x\to\infty} \sum_0^\infty \frac{(-1)^n}{n!} e^{n^2/2} \int_{n-x}^\infty e^{-y^2/2} (1-e^{-y/2}) dy = (1-e^{1/8}) \int e^{-y^2} e^{-e^{-y}} dy.$$

The Fourier transform of k is $\hat{k}(u) = \sqrt{2\pi}\, e^{-u^2/2}(1-e^{1/8+iu/2})$ and $\hat{k}(w) \neq 0$ except for $w = \frac{i}{4} + 4\pi m$, $m \in \mathbb{Z}$. The hypotheses on the kernel k are now easily checked to be satisfied except for (2.3.1), where we have

$$\left| \frac{\hat{k}_\alpha(u+\xi)}{\hat{k}_\alpha(u)} \right| \geq \frac{1-e^{1/8-\alpha/2}}{1+e^{1/8-\alpha/2}} e^{-\xi u - \xi^2/2}, \ \alpha > \frac{1}{4},$$

i.e. $\Theta(u) = |u|$.

The question whether the separation (2.1.2) of the sequence $\{\lambda_n\}$ is necessary, will be answered in the next theorem.

Theorem 2.13. *Let* $k \in L^1(\mathbb{R})$ *and let the sequence* $\{\lambda_n\}$ *satisfy*

$$0 < \lambda_1 < \lambda_2 \dots < \lambda_n \to \infty, \ n \to \infty$$

and

$$\liminf_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0.$$

Then there exist coefficients $\{a_n\}$ such that

$$F(x) = \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy$$

converges uniformly on every set $\{x: x < x_0\}$, $x_0 < \infty$, and

$$\lim_{x \to \infty} F(x) = 0,$$

but $\quad a_n \not\to 0, \; n \to \infty.$

Proof. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two disjoint subsequences of $\{\lambda_n\}$ such that

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots$$

and

$$\Sigma(\beta_n - \alpha_n) < \infty.$$

Define

$$F(x) = \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy,$$

where
$$a_n = \begin{cases} 1 & \text{if} \quad \lambda_n \in \{\alpha_n\} \\ -1 & \text{if} \quad \lambda_n \in \{\beta_n\} \\ 0 & \text{elsewhere} \end{cases} .$$

Thus we have

$$F(x) = \Sigma\left(\int_{\alpha_n - x}^{\infty} - \int_{\beta_n - x}^{\beta}\right)k(y)dy = \Sigma \int_{\alpha_n - x}^{\beta_n - x} k(y)dy.$$

Let $\varepsilon > 0$ and take N such that $\sum_{N+1}^{\infty} (\beta_n - \alpha_n) < \varepsilon$. Hence we get

$$\limsup_{x \to \infty} |F(x)| \leq \limsup_{x \to \infty} \sum_{1}^{N} \int_{\alpha_n - x}^{\beta_n - x} |k(y)|dy +$$

$$+ \limsup_{x \to \infty} \sum_{N+1}^{\infty} \int_{\alpha_n}^{\beta_n} |k(y-x)|dy \leq \max_{|M| = \varepsilon} \int_{M} |k(y)|dy$$

i.e. $\quad \lim_{x \to \infty} F(x) = 0.$

We finally remark that the conditions of regularity we have imposed on k can be relaxed, if the extension of the Fourier transform $\hat{k}$ has a zero free strip above the real axis.

Chapter 3.

3.1. Introduction.

Let

(3.1.1) $$F(x) = \Sigma\, a_n \int_{\lambda_n - x}^{\infty} k(y)\, dy \quad , \quad \lambda_{n+1} - \lambda_n \geq c > 0 \;.$$

If $k \in \mathcal{S}$ and $\{a_n\} \in l_p$ in (3.1.1), then it is readily seen that $F' \in L^p(\mathbb{R})$. On the other hand, if F defined by (3.1.1) has a derivative $F' \in L^p(\mathbb{R})$, does it follow that $\{a_n\} \in l_p$?

In the case $k(x) = \exp(x - \exp x)$, the theorems C and D answer this question affirmatively.

In this chapter we will study the above mentioned problem for general kernels.

3.2. A basic lemma.

The following lemma will be an essential part in the proof of a generalization of the theorems C and D.

Lemma 3.1. *Suppose that* $k \in \mathcal{S}$ *and that its Fourier transform* $\hat{k}$ *satisfies*

(3.2.1) $$\hat{k}(u) \neq 0 \quad , \quad u \in \mathbb{R} \;.$$

Let $\{\lambda_n\}$ *be a sequence of positive numbers such that*

(3.2.2) $$\lambda_{n+1} - \lambda_n \geq c > 0 \;.$$

Suppose further that there exists a finite constant N *such that*

(3.2.3) $$a_n = O(\lambda_n^N) \;.$$

Let

$$F(x) = \Sigma\, a_n \int_{\lambda_n - x}^{\infty} k(y) dy \;.$$

<u>Then</u>

(3.2.4) $\qquad F' \in L^p(\mathbb{R})\ ,\quad 1 \leq p < \infty\ ,$

<u>implies that</u>

(3.2.5) $\qquad \Sigma\ |a_n|^p \leq A \int |F'(x)|^p\ dx\ ,$

<u>where</u> A <u>is a constant depending only on</u> c, p <u>and the kernel</u> k.

<u>Proof</u>. Let $\mu = \Sigma\ a_n\ \tau_{-\lambda_n}\delta$. Then we can write

$$F'(x) = \mu * k(-x)\ ,$$

where $\mu \in \mathcal{S}'$, because of (3.2.2) and (3.2.3).

Let $\{H_n\}$ be the functions constructed in lemma 1.2 with $\mathrm{supp}(\hat{H}_n) \subset (-\beta,\beta)$, $\beta > \pi/c$.

Since $\hat{k}(u) \neq 0$ for $u \in \mathbb{R}$, it exists a $\psi \in \mathcal{S}$ such that

$$\frac{1}{\hat{k}(u)} = \hat{\psi}(u) \quad \text{for} \quad u \in [-\beta,\beta]\ .$$

Hence,

(3.2.6) $$\frac{\hat{H}_n(u)}{\hat{k}(u)} = (H_n * \psi)\hat{\ }(u)\ .$$

Moreover,

(3.2.7) $$\|H_n * \psi\|_{L^q} \leq \|\psi\|_{L^q}\,\|H_n\|_{L^1} \leq C_1\ .$$

From (3.2.4), (3.2.6) and (3.2.7) it follows that

$$(F')^{\vee} * (H_n * \psi) \in L^\infty(\mathbb{R})$$

and

$$[(F')^{\vee} * (H_n * \psi)]\hat{\ } = \hat{H}_n \cdot \hat{\psi} \cdot (\mu * k)\hat{\ } = (\mu * H_n)\hat{\ } \quad \text{in} \quad \mathcal{S}'.$$

Thus,

(3.2.8) $$|a_n| = |\Sigma a_k H_n(\lambda_k)| = |\mu * H_n(0)| \le C_1 \cdot \|F'\|_{L^p} .$$

Define a function

$$q(x) = \begin{cases} (6+c)/c & \text{for } 0 < x < c \\ 0 & \text{elsewhere} \end{cases} .$$

This function can, because of (3.2.1), be approximated, in the sense of Wiener, by translates of the function k in L^1-norm, i.e. for each $\varepsilon > 0$ there exist sequences $\{\alpha_i\}_1^N$ and $\{\mu_i\}_1^N$ such that

(3.2.9) $$\int \left| \sum_1^N \alpha_i k(\mu_i - x) - q(x) \right| dx < \varepsilon .$$

Putting $f(x) = \sum_1^N \alpha_i k(\mu_i - x)$, we have

(3.2.10) $$\left(\int_{-\infty}^0 + \int_c^\infty \right) |f(x)| \, dx < \varepsilon$$

Also, there exists a subset $M \subset (0, c)$ with $|M| \ge c/2$ and

(3.2.11) $$|f(x)| \ge (4+c)/c \quad \text{for} \quad x \in M .$$

If this is not true, then

$$\int_{(0,c)\setminus M} |f(x)-q(x)| dx \ge \int_{(0,c)\setminus M} (q(x)-|f(x)|) \, dx >$$

$$> \left(\frac{6+c}{c} - \frac{4+c}{c} \right) \cdot \frac{c}{2} = 1 ,$$

which contradicts (3.2.9) for $\varepsilon < 1$.

In particular, (3.2.11) implies that

(3.2.12) $$\int_0^c |f(x)| \, dx \ge 1 .$$

Forming

$$G(x) = \sum_{1}^{N} \alpha_i F'(x-\mu_i) , \tag{3.2.13}$$

we get an upper estimate for the function G in L^p-norm by Minkowski's inequality

$$\left(\int |G(x)|^p dx \right)^{1/p} \leq \sum_{1}^{N} |\alpha_i| \left(\int |F'(x)|^p dx \right)^{1/p} . \tag{3.2.14}$$

The function in (3.2.13) can also be written as

$$G(x) = \Sigma\, a_n f(x-\lambda_n)$$

and we get the inequality

$$|G(x)| \geq |a_k| \; |f(x-\lambda_k)| - \sum_{n \neq k} |a_n| \; |f(x-\lambda_n)| . \tag{3.2.15}$$

Let $\{I_n\}$ be a sequence of intervals defined by $I_n = (\lambda_n, \lambda_n + c)$, $n = 1,2, \ldots$. Because of (3.2.2), these intervals are pairwise disjoint. If we integrate the inequality in (3.2.15) over the interval I_k and use (3.2.8), (3.2.10), (3.2.12), then

$$\int_{I_k} |G(x)| \, dx \geq |a_k| \int_{I_k} |f(x-\lambda_k)| \, dx - \sum_{n \neq k} |a_n| \int_{I_k} |f(x-\lambda_n)| \, dx \geq$$

$$\geq |a_k| - C_2 \cdot \sum_{n \neq k} \int_{I_k} |f(x-\lambda_n)| \, dx \geq |a_k| - C_2 \, \varepsilon .$$

Recalling (3.2.14), we see that

$$\limsup_{k \to \infty} |a_k| \leq 0 \tag{3.2.16}$$

Applying Jensen's inequality to (3.2.15), we get

$$|a_k|^p \; |f(x-\lambda_k)|^p \leq (|G(x)| + \sum_{n \neq k} |a_n| \; |f(x-\lambda_n)|)^p \leq \tag{3.2.17}$$

$$\leq 2^{p-1} \left(|G(x)|^p + (\sum_{n \neq k} |a_n| \; |f(x-\lambda_n)|)^p \right) .$$

Put $J_k(x) = \sum_{n \neq k} |f(x-\lambda_n)|$ and apply Jensen's inequality once more to get

(3.2.18) $$\left(\sum_{n\neq k} |a_n||f(x-\lambda_n)|\right)^p \leq (J_k(x))^{p-1} \sum_{n\neq k} |a_n|^p |f(x-\lambda_n)|.$$

Inserting (3.2.18) in (3.2.17), we find that

(3.2.19) $$2^{p-1}|G(x)|^p \geq |a_k|^p |f(x-\lambda_k)|^p - 2^{p-1} \sum_{n\neq k} |a_n|^p (J_k(x))^{p-1} |f(x-\lambda_n)|.$$

Let $N_k = \{x \in I_k : J_k(x) > 1\}$. The Lebesgue measure of N_k is rather small as the following inequality shows

$$|N_k| \leq \int_{N_k} J_k(x)dx \leq \int_{I_k} J_k(x)dx = \sum_{n\neq k} \int_{I_k} |f(x-\lambda_n)|dx < \varepsilon.$$

If the inequality in (3.2.19) is integrated over the set $I_k \setminus N_k$, then

(3.2.20) $$2^{p-1} \int_{I_k\setminus N_k} |G(x)|^p dx \geq |a_k|^p \int_{I_k \setminus N_k} |f(x-\lambda_k)|^p dx -$$

$$- 2^{p-1} \sum_{n\neq k} |a_n|^p \int_{I_k\setminus N_k} (J_k(x))^{p-1} |f(x-\lambda_n)|dx \geq$$

$$\geq |a_k|^p \int_{I_k\setminus N_k} |f(x-\lambda_k)|^p dx - 2^{p-1} \sum_{n\neq k} |a_n|^p \int_{I_k\setminus N_k} |f(x-\lambda_n)|dx.$$

We know from (3.2.11) that there exists a subset M_k of I_k, with $|M_k| \geq c/2$, such that $|f(x-\lambda_k)| \geq (4+c)/c$ if $x \in M_k$. Hence,

(3.2.21) $$\int_{I_k\setminus N_k} |f(x-\lambda_k)|^p dx \geq \int_{M_k\setminus N_k} |f(x-\lambda_k)|^p dx \geq \int_{M_k\setminus N_k} |f(x-\lambda_k)|dx \geq$$

$$\geq \frac{4+c}{c}(|M_k|-|N_k|) \geq \frac{4+c}{c} \cdot \frac{c}{4} > 1 \quad \text{if} \quad \varepsilon < c/4.$$

Combining (3.2.20) and (3.2.21), we get

(3.2.22) $$2^{p-1} \int_{I_k} |G(x)|^p dx \geq |a_k|^p - 2^{p-1} \sum_{n\neq k} |a_n|^p \int_{I_k} |f(x-\lambda_n)|dx.$$

Since (3.2.16) holds, the sequence $\{a_n\}$ can be rearranged to a new sequence $\{a_{n_k}\}$, where the coefficients satisfy

$$|a_{n_1}| \geq |a_{n_2}| \geq |a_{n_3}| \geq \cdots\cdots .$$

From (3.2.22) it follows that

(3.2.23) $$2^{p-1}\int |G(x)|^p dx \geq 2^{p-1}\sum_{k=1}^{M}\int_{I_{n_k}} |G(x)|^p dx \geq$$

$$\geq \sum_{k=1}^{M}|a_{n_k}|^p - 2^{p-1}\sum_{k=1}^{M}\sum_{n\neq n_k}|a_n|^p\int_{I_{n_k}}|f(x-\lambda_n)|\,dx.$$

The second term to the right in (3.2.23) can be estimated by

$$\sum_{k=1}^{M}\sum_{n\neq n_k}|a_n|^p\int_{I_{n_k}}|f(x-\lambda_n)|dx = \sum_{k=1}^{M}\sum_{\substack{l=1\\ l\neq k}}^{M}|a_{n_l}|^p\int_{I_{n_k}}|f(x-\lambda_{n_l})|\,dx +$$

$$+\sum_{k=1}^{M}\sum_{n\notin\{n_k\}_1^M}|a_n|^p\int_{I_{n_k}}|f(x-\lambda_n)|\,dx \leq$$

$$\leq \sum_{l=1}^{M}|a_{n_l}|^p\sum_{\substack{k=1\\ k\neq l}}^{M}\int_{I_{n_k}}|f(x-\lambda_{n_l})|\,dx + |a_{n_M}|^p\sum_{k=1}^{M}\sum_{n\notin\{n_k\}_1^M}\int_{I_{n_k}}|f(x-\lambda_n)|\,dx$$

$$\leq \varepsilon\sum_{l=1}^{M}|a_{n_l}|^p + \varepsilon M|a_{n_M}|^p \leq 2\varepsilon\sum_{k=1}^{M}|a_{n_k}|^p.$$

Thus, for every positive integer M, we have

(3.2.24) $$2^{p-1}\int |G(x)|^p dx \geq (1-2^p\cdot\varepsilon)\sum_{k=1}^{M}|a_{n_k}|^p.$$

Let $\varepsilon < \min(c/4,\ 2^{-p})$. Then (3.2.24) implies that

(3.2.25) $$\int |G(x)|^p dx \geq A\sum |a_n|^p.$$

Combining (3.2.14) and (3.2.25), we get the conclusion (3.2.5).

3.3. The main result.

By the aid of lemma 3.1 we can now prove a generalization of the theorems C and D. This result is formulated in the next theorem.

Theorem 3.2. *Let* $\{\lambda_n\}$ *be a sequence of positive numbers with maximal density* D *and*

(3.3.1) $$\lambda_{n+1} - \lambda_n \geq c > 0.$$

Suppose that there exists an $\alpha > 0$ *such that*

(3.3.2) $$e^{-\alpha x}\,\Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy$$

converges uniformly on every set $\{x: x < x_0\}$, $x_0 < \infty$, and defines a bounded function on $\mathbb{R}$.

Suppose further that

$$k \in \mathcal{L}_\alpha \cap \mathcal{S},$$

$$\hat{k}(\cdot + iv) \in L^2(\mathbb{R}), \quad 0 < v < v_0, \; v_0 > \alpha,$$

on every line $\operatorname{Im} w = \sigma$, $0 < \sigma < \alpha$, there exists a closed interval I with $|I| = 2\pi D$ and $\hat{k}|_I \not\equiv 0$, and

$$\hat{k}(u) \not\equiv 0, \; u \in \mathbb{R}.$$

Let

$$F(x) = \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy.$$

Then

(3.3.3) $$F' \in L^p(\mathbb{R}), \quad 1 \leq p < \infty,$$

implies that

(3.3.4) $$\Sigma |a_n|^p \leq A \int |F'(x)|^p dx,$$

where A is a constant depending only on c, p and the kernel k.

Proof. The hypotheses (3.3.1) and (3.3.2) together with $k \in \mathcal{L}_\alpha$ imply, by theorem 2. 9 , that

$$a_n = O(e^{\alpha \lambda_n}).$$

Since $F \in \mathcal{S}'$, follows from (3.3.3), lemma 2.4 gives

$$F(x) = \int_{-x}^{\infty} g(x)dt,$$

where

$$\hat{g}(w) = \hat{k}(w)h(w) \text{ is analytic for } \operatorname{Im} w > 0.$$

From this and the assumptions on $\hat{k}$, we find, just as in the proof of theorem 2.1, that

$$g(x) = \Sigma a_n \, k(x+\lambda_n), \; a_n = O(\lambda_n^N), \quad N \text{ finite constant.}$$

Now lemma 3.1 can be applied to get (3.3.4) and the theorem is proved.

Remark about the L^1-case: Suppose that $k \in \mathcal{L}_\alpha$, $\alpha = 0$. Then there is no need for the hypothesis $\hat{k}(\cdot + iv) \in L^2(\mathbb{R})$ in the theorem. We don't even need that $k \in \mathcal{S}$.

Specialization of theorem 3.2 gives a more general theorem than Waterman's in [18]. We state this in the next corollary.

Corollary 3.3. *Let*

$$f(t) = \Sigma a_n \, e^{-p_n t}, \; t > 0 \;,$$

be a lacunary series, i.e. $p_{n+1}/p_n \geq q > 1$, *and* $p_1 > 0$. *Let* $1 \leq p < \infty$, $\beta > 0$, *and suppose that*

$$\int_0^\infty t^{\beta-1} |f'(t)|^p dt < \infty. \tag{3.3.5}$$

Then

$$\Sigma |a_n|^p \, p_n^{\,p-\beta} \leq A \int_0^\infty t^{\beta-1} |f'(t)|^p dt,$$

where the constant A *depends on* q, β *and* p.

Proof. The integral in (3.3.5) can also be written as

$$\int_0^\infty t^{\beta-1} |f'(t)|^p dt = \int_{-\infty}^\infty |f'(e^{-x}) e^{-x \, \beta/p}|^p \, dx.$$

Define a function F by

$$F'(x) = f'(e^{-x}) e^{-x \, \beta/p} \quad \text{and} \quad F(-\infty) = 0.$$

Thus we have that

$$F'(x) = - \Sigma a_n \, e^{(1-\beta/p)\log p_n} \; k(\log p_n - x), \tag{3.3.6}$$

where $k(x) = \exp(\frac{\beta}{p}x - \exp x)$. The Fourier transform of k is

$$\hat{k}(u) = \Gamma(\beta/p - iu). \tag{3.3.7}$$

From (3.3.7) it follows easily that $k \in \mathcal{L}_\alpha$, $\alpha \geq 0$. The other hypotheses on k in theorem 3.2 are also obvious from the knowledge of the Γ-function. (See

for instance [1]). The assumptions (3.3.1) and (3.3.2) follow immediately and we get from (3.3.6) that

$$\Sigma |a_n e^{(1-\beta/p)\log p_n}|^p \le A \int_{-\infty}^{\infty} |F'(x)|^p dx.$$

Hence,

$$\Sigma |a_n|^p \, p_n^{p-\beta} \le A \int_0^{\infty} t^{\beta-1}|f'(t)|^p \, dt$$

and the corollary is proved.

3.4. Hausdorff-Young's inequality for Dirichlet series.

In this section we prove an inequality for Dirichlet series, which can be used to obtain a stronger result than that in lemma 3.1 in the case $p = 2$.

Theorem 3.4. Let

$$f(w) = \Sigma \alpha_n e^{iw\lambda_n} \quad \text{for} \quad \operatorname{Im} w > 0,$$

where $\{\lambda_n\}$ is a sequence of positive numbers satisfying

(3.4.1) $$\lambda_{n+1} - \lambda_n \ge c > 0.$$

Let $1 \le p \le 2$ and suppose that

(3.4.2) $$f(\cdot+iv) \to \tilde{f} \in L^p(I) \quad \text{in} \quad \mathcal{D}'(I),\ v \downarrow 0,$$

where I is an open interval with $|I| > 2\pi/c$.

Then there exists a constant A, depending on c, p and $|I|$, such that

(3.4.3) $$\{\Sigma |\alpha_n|^q\}^{1/q} \le A \,\|\tilde{f}\|_{L^p(I)}, \qquad \left(\frac{1}{q} + \frac{1}{p} = 1\right).$$

Proof. It is no loss in generality to assume that $I = (-\pi/c-\varepsilon, \pi/c+\varepsilon)$ for some $\varepsilon > 0$. Let $\hat{\phi} \in \mathcal{D}(I)$ and apply (3.4.2), we get

(3.4.4) $$\lim_{v\downarrow 0} |\Sigma \alpha_n e^{-v\lambda_n} \frac{1}{2\pi} \phi(\lambda_n)| = \lim_{v\downarrow 0} |\int f(u+iv)\, \hat{\phi}(u)du| =$$

$$= |\int \tilde{f}(u)\, \hat{\phi}(u)du| \le \|\tilde{f}\|_{L^p(I)} \|\hat{\phi}\|_{L^q(\mathbb{R})}.$$

If ϕ is chosen to be the function H_k from lemma 1.2, then (3.4.4) implies that $\alpha_n = O(1)$.

Hence we find that

$$|\Sigma \alpha_n \phi(\lambda_n)| \le 2\pi \|\tilde{f}\|_{L^p(I)} \|\hat{\phi}\|_{L^q(\mathbb{R})}.$$

Thus, if $\phi = \sum_1^N \beta_k H_k$, then

(3.4.5) $$|\sum_1^N \alpha_n \beta_n| = |\Sigma \alpha_n \sum_{k=1}^N \beta_k H_k(\lambda_n)| \le 2\pi \|\tilde{f}\|_{L^p(I)} \|\sum_1^N \beta_k \hat{H}_k\|_{L^q(\mathbb{R})} \le$$

$$\le C \|\tilde{f}\|_{L^p(I)} \|\sum_1^N \beta_k H_k\|_{L^p(\mathbb{R})}.$$

The last inequality is given by Hausdorff-Young's inequality on the real line. Moreover,

$$\int |\sum_1^N \beta_k H_k(x)|^p dx \le \int (\Sigma H_j(x))^{p-1} \sum_1^N |\beta_k|^p |H_k(x)| dx \le$$

$$\le C_1^{p-1} \sum_1^N |\beta_k|^p \int |H_k(x)| dx,$$

and we get from (3.4.5) that

(3.4.6) $$|\sum_1^N \alpha_n \beta_n| \le C_p \|\tilde{f}\|_{L^p(I)} (\sum_1^N |\beta_k|^p)^{1/p}.$$

Since sequences with a finite number of non-zero entries are dense in l_p, the condition (3.4.6) implies (3.4.3) and proves the theorem.

The hypothesis (3.4.1) can not be weakened as the next example shows.

<u>Example</u>. Let $\alpha_n = (-1)^{n+1} n^{-1/q}$ $(\frac{1}{q} + \frac{1}{p} = 1,\ p > 1)$ and $\lambda_n = \log n$. Define

$$f(w) = \sum_{n=1}^{\infty} \alpha_n e^{iw \log n} = \sum_1^{\infty} \frac{(-1)^{n+1}}{n^{1/q - iw}} = \eta(\frac{1}{q} - iw) \quad \text{for} \quad \operatorname{Im} w > 0.$$

Since $f(w)$ has an analytic extension into the whole plane, we have in particular

$$f(\cdot + iv) \to f \quad \text{in} \quad L^p_{loc}(\mathbb{R}),\ v \downarrow 0,$$

but $\Sigma |\alpha_n|^q$ is divergent.

We also give an inequality in the other direction and formulate this in the following theorem.

Theorem 3.5. Let $\{\lambda_n\}$ be a sequence of positive numbers satisfying

(3.4.6) $$\lambda_{n+1} - \lambda_n \geq c > 0.$$

Let $1 \leq p \leq 2$ and let I be a finite interval on $\mathbb{R}$. If $\{\alpha_n\} \in l_p$, then there exist a function $\tilde{f} \in L^q(I)$ and a constant A, depending on p, $|I|$ and c, such that

$$f(\cdot+iv) \to \tilde{f} \quad \text{in} \quad L^q(I), \quad v \downarrow 0,$$

and

(3.4.7) $$\|\tilde{f}\|_{L^q(I)} \leq A\,\{\Sigma|\alpha_n|^p\}^{1/p}, \quad (\frac{1}{p} + \frac{1}{q} = 1)$$

where

$$f(u+iv) = \Sigma\alpha_n \; e^{i(u+iv)\lambda_n}, \quad v > 0.$$

Proof. For $p = 1$ the result is obvious. Suppose that $1 < p \leq 2$. Let $\hat{\phi} \in \mathcal{D}$ and $\hat{\phi}|_I = 1$. Define measures

$$\mu_v = \Sigma\alpha_n \; e^{-v\lambda_n} \; \tau_{-\lambda_n}\delta \;, \; v \geq 0 \;.$$

We have that

$$(\mu_v * \phi)\hat{} = \hat{\phi}\cdot\hat{\mu}_v = \hat{\phi}\cdot f(\cdot+iv) \quad \text{in } \mathcal{S}', \; v > 0.$$

Using Hausdorff-Young's inequality on the real line, we get

(3.4.8) $$\{\int_I |\hat{\mu}_{v_1}(u) - \hat{\mu}_{v_2}(u)|^q du\}^{1/q} \leq \{\int |(\hat{\mu}_{v_1}(u) - \hat{\mu}_{v_2}(u))\,\hat{\phi}(u)|^q du|^{1/q} \leq$$

$$\leq C\{\int |\mu_{v_1} * \phi(x) - \mu_{v_2} * \phi(x)|^p dx\}^{1/p} =$$

$$= C\,\{\int |\Sigma\,\alpha_n(e^{-v_1\lambda n} - e^{-v_2\lambda n})\phi(x+\lambda_n)|^p dx\}^{1/p} \leq$$

$$\leq C\{\int (\sum_{k=1}^{\infty} |\phi(x+\lambda_k)|)^{p-1}\,\Sigma|\alpha_n(e^{-v_1\lambda n} - e^{-v_2\lambda n})|^p |\phi(x+\lambda_n)|dx\}^{1/p}$$

$$\leq C_p\{\Sigma|\alpha_n(e^{-v_1\lambda n} - e^{-v_2\lambda n})|^p\}^{1/p}.$$

The last term tends to zero as $v_1, v_2 \to 0$, by dominated convergence. Thus,

there exists a function $\tilde{f} \in L^q(I)$ such that

$$\hat{\mu}_v \to \tilde{f} \quad \text{in} \quad L^q(I), \quad v \downarrow 0.$$

Moreover, from (3.4.8) follows also (3.4.7).

The assumption (3.4.6) is essential in the case $1 < p \leq 2$ as the following example shows.

Example. Let $\alpha_n = 1/n$ and $\lambda_n = \log n$. Define

$$f(w) = \sum_1^\infty \alpha_n e^{iw\lambda_n} = \zeta(1-iw) \quad \text{for} \quad \operatorname{Im} w > 0.$$

Here we have $\{\alpha_n\} \in l_p$, $1 < p \leq 2$, but the function $f(w)$ has an analytic extension into the whole plane except for a simple pole at $w = 0$.

By the aid of theorem 3.4 we can prove a stronger L^2-result than that in lemma 3.1. The assumption (3.2.1) is now replaced by

$$\hat{k}(u) \neq 0 \quad \text{if} \quad u \in \tilde{I}, \tag{3.4.9}$$

where $\tilde{I}$ is a closed interval and $|\tilde{I}| = 2\pi/c$.

Proof. The hypothesis (3.4.9) implies that there exists an open interval $I \supset \tilde{I}$ with $\hat{k}|_I \neq 0$. Let

$$g(x) = \mu * k(x) = F'(-x).$$

Hence,

$$\hat{g} = \hat{k} \cdot \hat{\mu} \quad \text{in } \mathscr{S}'. \tag{3.4.10}$$

Since $g \in L^2(\mathbb{R})$, we have

$$|\hat{g}(\phi)| = |g(\hat{\phi})| \leq 2\pi \| g \|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})}, \quad \phi \in \mathscr{D}(I). \tag{3.4.11}$$

From (3.4.10) and (3.4.11) it follows that

$$|\hat{\mu}(\phi)| = |\hat{g}(\phi/\hat{k})| \leq 2\pi \| g \|_{L^2(\mathbb{R})} \|\phi/\hat{k}\|_{L^2(\mathbb{R})} \leq$$

$$\leq C_k \| g \|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})}, \qquad \phi \in \mathscr{D}(I).$$

Let $\hat{\mu}_v(u) = \Sigma a_n e^{i(u+iv)\lambda_n}$, $v > 0$. Then is easily seen that

$$\hat{\mu}_v \to \hat{\mu} \quad \text{in } \mathcal{D}'(I), \quad v \downarrow 0.$$

Now theorem 3.4. can be applied to get

$$\Sigma |a_n|^2 \leq A \int |g(x)|^2 dx = A \int |F'(x)|^2 dx,$$

where the constant A depends on k, c and $|I|$.

Chapter 4

4.1. Introduction.

In this chapter we will estimate the remainder in the general high indices theorem, i.e. we look at the following problem:

Let $\gamma \in \mathbb{R}$ and suppose that

$$\Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy, \quad \lambda_{n+1} - \lambda_n \geq c > 0,$$

converges and defines a function $F(x)$, $x \in \mathbb{R}$, such that

$$F(x) = O(e^{\gamma x}), \quad x \to \infty.$$

What can then be said about the coefficients $\{a_n\}$?

Theorem 2.9 solves this problem partially for a lot of kernels. Among these are $k(x) = \exp(x - \exp x)$, i.e. we find that, if $\gamma \geq 0$ and $\{p_n\}$ is a Hadamard sequence $(p_1 > 0)$, then

$$\Sigma a_n e^{-p_n t} = O(t^{-\gamma}), \quad t \to +0,$$

implies that

$$a_n = O(p_n^{\gamma}).$$

This estimate for the coefficients $\{a_n\}$ is also the best possible result as the next example shows.

Example. Let $a_n = e^{\gamma\lambda_n}$, $\gamma > 0$, and define

$$F(x) = \Sigma a_n \int_{\lambda_n - x}^{\infty} e^{-e^y + y} dy = \Sigma e^{\gamma\lambda_n} e^{-e^{-(x-\lambda_n)}}.$$

Then, as is easily seen, we have

$$e^{-\gamma x} F(x) = \Sigma \exp(-\exp(-(x-\lambda_n)) - \gamma(x-\lambda_n)) = O(1), \quad x \in \mathbb{R}.$$

We treat the cases $\gamma \geq 0$ and $\gamma < 0$ separately and call these case I and case II respectively.

4.2 The case I.

In this section we restrict ourselves to the case $\gamma \geq 0$, and prove an extension of the result in theorem 2.9.

Theorem 4.1. *Let* $\{\lambda_n\}$ *be a sequence of positive numbers, with maximal density* D, *satisfying*

$$\lambda_{n+1} - \lambda_n \geq c > 0.$$

Suppose that there exists an $\alpha \geq 0$ *such that* $k \in \mathcal{L}_\alpha$ *and that*

$$e^{-\alpha x} \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy \tag{4.2.1}$$

converges uniformly on every set $\{x: x < x_0\}$, $x_0 < \infty$, *and defines a bounded function on* $\mathbb{R}$.

Let $0 \leq \gamma \leq \alpha$, *and suppose that*

$$k_\gamma \in \mathcal{S}, \tag{4.2.2}$$

$$k_{v_0} \in L^2(\mathbb{R}) \text{ for some } v_0 > \alpha, \tag{4.2.3}$$

(4.2.4) *on every line* $\operatorname{Im} w = \sigma$, $\gamma \leq \sigma < \alpha$, *there exists a closed interval* I *such that* $\hat{k}(u+i\sigma) \neq 0$ *if* $u \in I$, *where* $|I| = 2\pi D$ *for* $\gamma < \sigma < \alpha$ *and* $|I| = 2\pi/c$ *for* $\sigma = \gamma$.

Then

$$\Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy = O(e^{\gamma x}) \text{ as } x \to \infty \tag{4.2.5}$$

implies that

$$a_n = O(e^{\gamma \lambda_n}). \tag{4.2.6}$$

In particular, if $\gamma = 0$ *and* $\hat{k}(0) \neq 0$, *then*

$$\sum_{n \leq x} a_n = O(1), \quad x \in \mathbb{R}. \tag{4.2.7}$$

Proof. From the assumption (4.2.1) and that $k \in \mathcal{L}_\alpha$, we get that $a_n = O(e^{\alpha \lambda_n})$, by using theorem 2.9.

Let

(4.2.8) $$F(x) = \Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)dy.$$

Because of the assumption (4.2.3) we get from lemma 2.2 that F can be written in the form

$$F(x) = \int_{-x}^{\infty} g(t)dt$$

where

(4.2.9) $$\hat{g}_{v_0} = \hat{k}(\cdot+iv_0)\ h(\cdot+iv_0) \in L^2(R).$$

Since $F(x)e^{-\gamma x}$ is in $\mathcal{S}'$, we also have $g(x)e^{\gamma x}$ in $\mathcal{S}'$, because

$$(F(x)e^{-\gamma x})' = F'(x)e^{-\gamma x} - \gamma F(x)e^{-\gamma x}.$$

Using (4.2.9) and lemma 2.3, we get

$$\hat{g}(u+iv) = \hat{k}(u+iv)\ h(u+iv) \text{ is analytic for } v > \gamma.$$

The hypotheses (4.2.2) and (4.2.4) now give, just as in the proof of theorem 2.1, that

(4.2.10) $$a_n e^{-\gamma\lambda n} = O(\lambda_n^N) \text{ for some finite constant } N.$$

Define a measure

(4.2.11) $$\mu = \Sigma a_n e^{-\gamma\lambda n}\ \tau_{-\lambda_n}\delta\ .$$

Consider first the case $\gamma > 0$:

The function F in (4.2.8) can also be written as

$$F(x) = e^{\gamma x}\Sigma a_n e^{-\gamma\lambda n}\ \chi(\lambda_n - x) = e^{\gamma x}\ \mu * \chi(-x),$$

where

$$\hat{\chi}(u) = -\frac{k_\gamma(u)}{i(u+i\gamma)}\ .$$

Since $k_\gamma \in \mathcal{S}$, it follows that $\chi \in \mathcal{S}$. The measure μ in (4.2.11) is in $\mathcal{S}'$,

because of (4.2.10). Without loss of generality we can assume that there exists an open interval $I \supset [-\pi/c-\varepsilon, \pi/c+\varepsilon]$ such that $\hat{k}_\gamma|_{\bar{I}} \not\equiv 0$. Let H_k be the function in lemma 1.2 with $\mathrm{supp}(\hat{H}_k) \subset I$, and let $\psi \in \mathcal{S}$ satisfy

$$\hat{\psi}(u) = \frac{1}{\hat{\chi}(u)}, \quad u \in I.$$

Hence, we get

$$((F_{-\gamma})^{\vee} * \psi * H_k)^{\wedge} = \hat{\psi}\cdot\hat{H}_k \cdot \hat{\chi}\,\hat{\mu} = (\mu * H_k)^{\wedge} \quad \text{in } \mathcal{S}',$$

and

$$|a_k e^{-\gamma\lambda_k}| = |\mu * H_k(0)| \leq \int |F_{-\gamma}(y)||\psi * H_k(y)|\,dy = O(1),$$

because of (4.2.5). Thus (4.2.6) is proved.

In the case $\gamma = 0$, the conclusions (4.2.6) and (4.2.7) respectively are obtained by theorem 1.3.

4.3. The case II.

By a similar technique as in the last section, we can prove a generalization of theorem F. In the statement we use $\gamma > 0$ and we have

Theorem 4.2. Suppose that there exists an $\alpha \geq 0$ such that $k \in \mathcal{L}_\alpha$ and that

(4.3.1) $$e^{-\alpha x}\,\Sigma a_n \int_{\lambda_n - x}^{\infty} k(y)\,dy$$

converges uniformly on every set $\{x: x < x_0\}$, $x_0 < \infty$, and defines a bounded function on $\mathbb{R}$. The sequence $\{\lambda_n\}$ consists of positive numbers, with maximal density D, satisfying

$$\lambda_{n+1} - \lambda_n \geq c > 0.$$

Let $\gamma > 0$, and suppose that

$$k_{v_0} \in L^2(\mathbb{R}) \quad \text{for some } v_0 > \alpha,$$

<u>the Fourier transform</u> $\hat{k}$ <u>has a meromorphic extension into the halfplane</u> $\operatorname{Im} w > -\gamma$,

<u>the boundary function</u> $\hat{k}(\cdot - i\gamma) \in \mathcal{S}$, <u>or at least</u> $(u_0 - \cdot)\, \hat{k}(\cdot - i\gamma) \in \mathcal{S}$ <u>for some</u> $u_0 \in \mathbb{R}$,

<u>on every line</u> $\operatorname{Im} w = \sigma$, $-\gamma \leq \sigma < \alpha$, <u>there exists a closed interval</u> I <u>such that</u> $\hat{k}(u+i\sigma) \not\equiv 0$ <u>for</u> $u \in I$, <u>where</u> $|I| = 2\pi D$ <u>if</u> $-\gamma < \sigma < \alpha$ <u>and</u> $|I| = 2\pi/c$ <u>if</u> $\sigma = -\gamma$.

<u>Let</u>

(4.3.2) $$\sum a_n \int_{\lambda_n - x}^{\infty} k(y)dy = O(e^{-\gamma x}) \quad \text{as} \quad x \to \infty.$$

<u>Then</u>

(4.3.3) $$a_n = O(e^{-\gamma\lambda_n}).$$

<u>Proof.</u> Let

$$F(x) = \sum a_n \int_{\lambda_n + x}^{\infty} k(y)dy.$$

Then, just as in the proof of theorem 4.2, we get

(4.3.4) $$F(x) = \int_x^{\infty} g(t)dt,$$

where

(4.3.5) $$\hat{g}(u+iv) = \hat{k}(u+iv)h(u+iv)$$

is analytic in $w = u+iv$ for $v > -\gamma$. We also get that $a_n e^{\gamma\lambda_n} = O(\lambda_n^N)$ for some finite constant N. From (4.3.4) and (4.3.5) it follows that

$$\hat{F}(u+iv) = -\frac{\hat{k}(u+iv)}{i(u+iv)}h(u+iv) \quad \text{for} \quad v > -\gamma.$$

f $\hat{k}(\cdot - i\gamma) \in \mathcal{S}$, then

$$F_{-\gamma}(x) = \mu * \chi(x),$$

here $\mu = \Sigma a_n e^{\gamma\lambda_n} \tau_{-\lambda_n} \delta$ is in $\mathcal{S}'$ and $\hat{\chi}(u) = -\dfrac{\hat{k}(u-i\gamma)}{i(u-i\gamma)}$ is in $\mathcal{S}$. Without

oss of generality we can assume that $I \supset [-\pi/c-\varepsilon, \pi/c + \varepsilon]$, $\varepsilon > 0$, and that

$(\cdot - i\gamma)|_I \neq 0$. Let $\psi \in \mathcal{S}$ with

$$\hat{\psi}(u) = \frac{1}{\hat{\chi}(u)}, \; u \in I.$$

rom (4.3.1) and (4.3.2) it follows that $F_{-\gamma}(x) = O(1)$ and, in particular,

$_{-\gamma} \in \mathcal{S}'$. Let H_k be the function in lemma 1.2. Then we get

$$(F_{-\gamma} * H_k * \psi)^\wedge = \hat{H}_k \cdot \hat{\psi} \cdot (\mu * \chi)^\wedge = (\mu * H_k)^\wedge \text{ in } \mathcal{S}'.$$

hus,

$$|a_k e^{\gamma\lambda_k}| = |\mu * H_k(0)| \leq \| F_{-\gamma}\|_{L^\infty} \| H_k\|_{L^1} \|\psi\|_{L^1} = O(1)$$

In the case $\hat{k}(\cdot - i\gamma)$ has a singularity, but $(\cdot - u_0)\hat{k}(\cdot - i\gamma) \in \mathcal{S}$ for

ome $u_0 \in \mathbb{R}$, we denote by χ the function in $\mathcal{S}$ with

$$\hat{\chi}(u) = -\frac{(u-u_0)\hat{k}(u-i\gamma)}{i(u-i\gamma)} .$$

e can assume that $u_0 = 0$. Then we have

$$(F_{-\gamma})^\wedge = \hat{\chi} \cdot (G * \mu)^\wedge \text{ in } \mathcal{S}'.$$

ence,

$$F_{-\gamma}(x) = \Sigma\, a_n e^{\gamma\lambda_n} \int_{\lambda_n - x}^{\infty} \chi(y)dy = O(1).$$

y theorem 1.3 we get $a_n e^{\gamma\lambda_n} = O(1)$ and (4.3.3) is proved.

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